Theoretische Physik 2 (Elektrodynamik)

Wintersemester 2016/17

Abgabe bis Freitag, 16.12.16, 12:00 neben PH 3218.

Übungsblatt Nr. 8 Dieses Blatt wird in den Übungen vom 19.12. - 23.12.16 besprochen.

Aufgabe 1:
Neumann Green function — 6 Punkte

Consider the Green function appropriate for Neumann boundary conditions for a volume V (with boundary denoted by S) between the concentric spherical surfaces defined by $r = a$ and $r = b$, $a < b$. We take the simple constraint

$$\frac{\partial G_N}{\partial n'}(\mathbf{x}, \mathbf{x}') = -\frac{4\pi}{A} \quad \text{for } \mathbf{x}' \text{ on } S\,,$$

where A is the total area of the boundary S. Then we have the solution

$$\phi(\mathbf{x}) = \langle\phi\rangle_S + \int_V \rho(\mathbf{x}')\, G_N(\mathbf{x}, \mathbf{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\phi(\mathbf{x}')}{\partial n'}\, G_N(\mathbf{x}, \mathbf{x}')\, da',$$

where $\langle\phi\rangle_S$ is the average value for the potential over the whole surface. Use an expansion in spherical harmonics of the form

$$G(\mathbf{x}, \mathbf{x}') = \sum_{l=0}^{\infty} g_l(r, r')\, P_l(\cos\gamma),$$

where $g_l(r, r') = r_<^l / r_>^{l+1} + f_l(r, r')$, $r_< = \min(|\mathbf{x}|, |\mathbf{x}'|)$ and $r_> = \max(|\mathbf{x}|, |\mathbf{x}'|)$.

(a) Show that for $l > 0$, the radial Green function has the symmetric form

$$g_l(r, r') = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{(b^{2l+1} - a^{2l+1})}\left[\frac{l+1}{l}(rr')^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1}\left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right)\right].$$

Hint: Recall

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma),$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$. Then the first term in the expansion of $g_l(r, r')$ (and hence the $G(\mathbf{x}, \mathbf{x}')$) corresponds to the source of the singular delta function. The remaining term in $G(\mathbf{x}, \mathbf{x}')$,

$$F(\mathbf{x}, \mathbf{x}') = \sum_l f_l(r, r') P_l(\cos\gamma)\,,$$

solves the source-free equation $\nabla_{x'}^2 F(\mathbf{x}, \mathbf{x}') = 0$ whose solution is of the form

$$f_l(r, r') = A_l r'^l + B_l \frac{1}{r'^{l+1}}.$$

Finally, our goal is to apply the boundary conditions to solve $g_l(r, r')$.

(b) Show that for $l = 0$,

$$g_0(r, r') = \frac{1}{r_>} - \left(\frac{a^2}{a^2 + b^2}\right)\frac{1}{r'} + f(r),$$

where $f(r)$ is arbitrary. Show explicitly that the solution $\phi(\mathbf{x})$ given above is independent of $f(r)$.

Aufgabe 2:
Solving the electrostatic potential using the Neuman Green function 4 Punkte

Apply the Neumann Green function of Problem 1 to the situation in which the normal electric field $E_r = -E_0 \cos\theta$ on the outer surface ($r = b$) and $E_r = 0$ on the inner surface($r = a$), where E_0 is constant.

(a) Show that the electrostatic potential inside the volume V is

$$\phi(\mathbf{x}) = E_0 \frac{r\cos\theta}{1 - p^3}\left(1 + \frac{a^3}{2r^3}\right),$$

where $p = a/b$. Find the components of the electric field,

$$E_r(r, \theta) = -E_0 \frac{\cos\theta}{1 - p^3}\left(1 - \frac{a^3}{r^3}\right),$$

$$E_\theta(r, \theta) = E_0 \frac{\sin\theta}{1 - p^3}\left(1 + \frac{a^3}{2r^3}\right).$$

(b) Calculate the Cartesian or cylindrical components of the field, E_z and E_ρ, where $\rho = \sqrt{x^2 + y^2}$.

Hint: Useful relations for special functions

$$P_l(\cos\gamma) = \frac{4\pi}{2l + 1}\sum_m Y_l^m(\Omega) Y_l^{m*}(\Omega'),$$

$$\cos\theta = \sqrt{\frac{4\pi}{3}} Y_1^0(\Omega),$$

where Ω and Ω' are the spherical angle of $\mathbf{x}$ and $\mathbf{x}'$, respectively.