5. Matter Power Spectrum

We can now make use of the evolution equations and of the initial conditions to derive cosmic solutions. Of special interest is the evolution of CDM, that only depends very little on the radiation perturbations. The solutions $\delta(k, \eta)$ can then be compared with the observed matter distribution in the Universe.

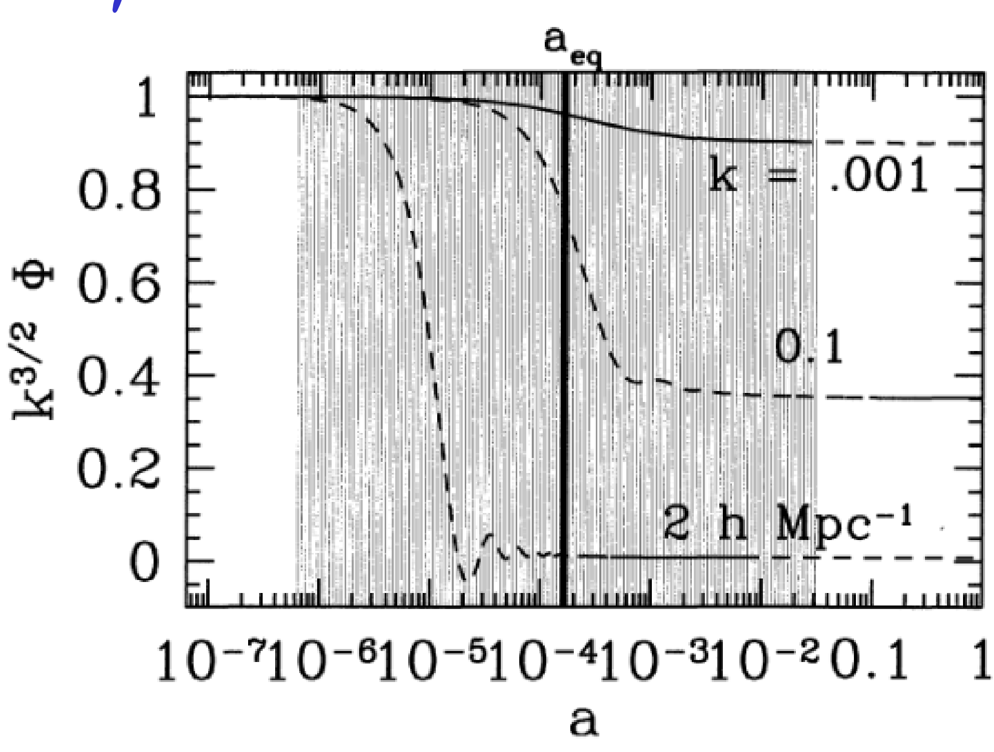
5.1 Overview

The concept of gravitational instability may be expressed in terms of the cartoon equation

$$\delta'' + [\text{Pressure} - \text{Gravity}]\,\delta = 0$$

Now, if pressure dominates, the solutions are oscillating, when gravity dominates, they decay exponentially. Unlike baryonic matter, CDM is pressureless, but the expanding background typically alters the simple picture presented in above equation.

We focus on the gravitational evolution of the potential Φ that is given for three different modes that enter the horizon after, during or before matter-radiation



equality at $a = a_{eq}$. Before horizon entry, the modes do not evolve, as discussed in the previous section. Then, there is a phase of k-dependent evolution: Modes that enter the horizon before equality experience substantial damping. Then, more deeply in the matter epoch, the growth of the modes proceeds universally again. This motivates a factorisation of the problem in the following manner:

$$\Phi(a,\vec{k}) = \Phi_p(\vec{k}) * \underbrace{T(\vec{k})}_{\text{transfer function}} * \frac{9}{10}\frac{D_1(a)}{a} \quad \rightarrow \text{growth function}$$

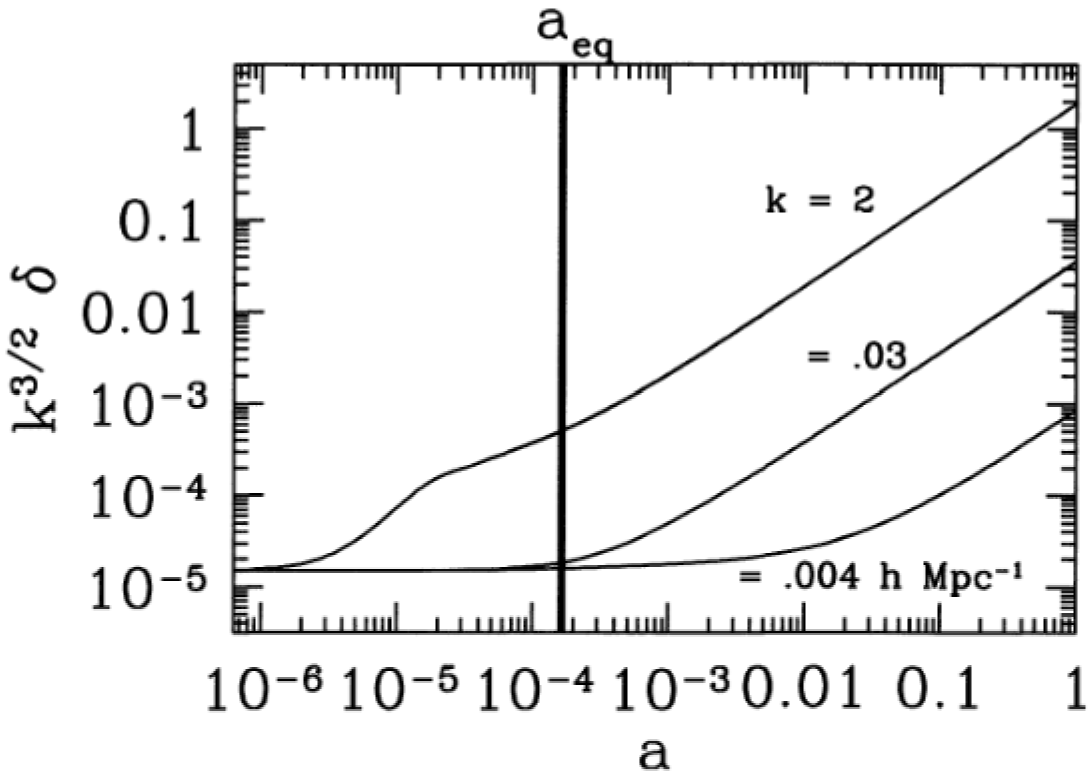
The regime governed by the transfer function is indicated as a shaded region in the figure.
We observe from the figure that even large-scale perturbations are suppressed (by a factor of $\frac{9}{10}$, as it will turn out). As we normalise T on large scales to be equal to one by defining it as

$$T(k) = \frac{\Phi(k, a_{late})}{\Phi_{large\text{-}scale}(k, a_{late})},$$

this incurs the extra factor $\frac{9}{10}$. The scale factor a_{late} should be chosen to be well after the transfer regime (i.e. to the right of the shaded region).
The ratio of the potential to its value after the transfer function regime defines the growth function $D_1(a)$ as

$$\frac{\Phi(a)}{\Phi(a_{late})} = \frac{D_1(a)}{a} \qquad (a > a_{late})$$

In a spatially flat, matter dominated Universe $D_1(a) = a$, such that Φ remains constant, as it appears also in the Figure. It turns out however, that the overdensity $\delta \propto a$, as it is illustrated by the following Figure:



Now, the poisson equation

$$k^2 \Phi = 4\pi G a^2 \rho_m \delta \quad (a > a_{late})$$

allows to construct the potential Φ at late times from the observed overdensities δ. The background matter density can be expressed as $\rho_m = \frac{\Omega_m}{a^3} \rho_{cr}$ where we recall that $4\pi G \rho_{cr} = \frac{3}{2} H_0^2$, such that

$$\delta(\vec{k}, a) = \frac{k^2 \Phi(\vec{k}, a)\, a}{\frac{3}{2} \Omega_m H_0^2} \quad (a > a_{late})$$

The relation between the overdensity today and the primordial potential is therefore given by

$$\delta(\vec{k}, a) = \frac{3}{5} \frac{k^2}{\Omega_m H_0^2} \Phi_p(\vec{k})\, T(k)\, D_1(a) \quad (a > a_{late})$$

The power spectrum is again defined as

$$\langle \delta(\vec{k},a)\,\delta(\vec{k}',a)\rangle = (2\pi)^3\, P(\vec{k},a)\, \delta^3(\vec{k}-\vec{k}')$$

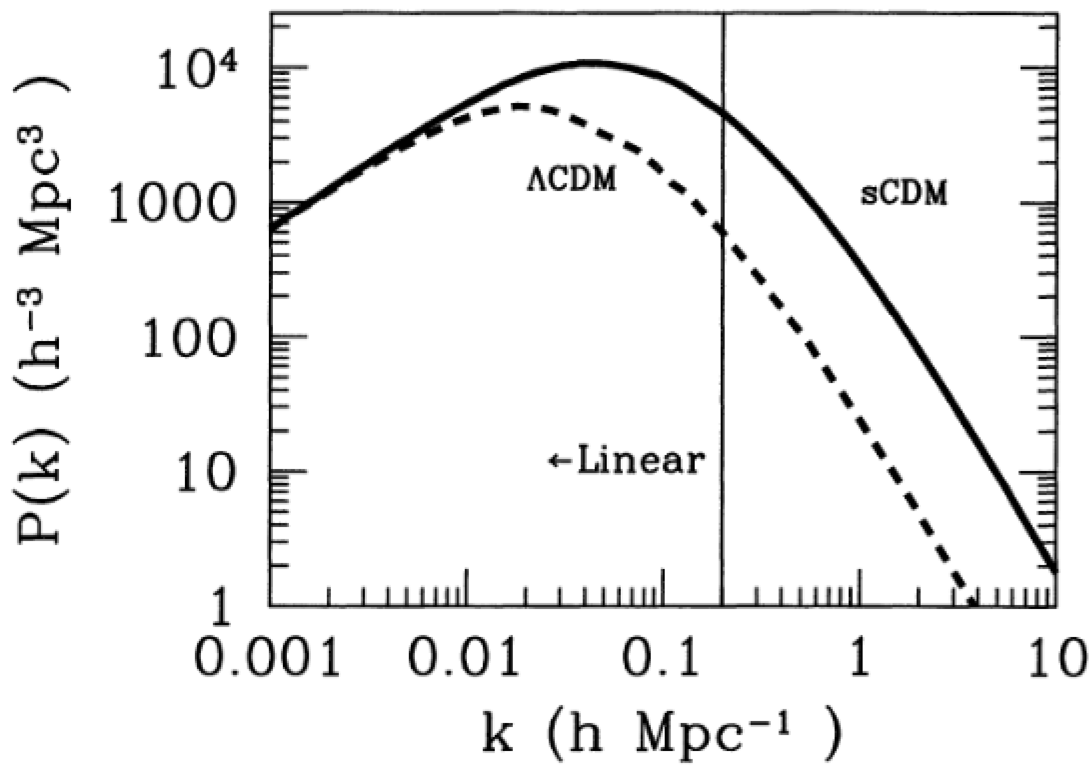
such that

$$P(\vec{k},a) = \frac{9}{25}\,\frac{k^4}{\Omega_m^2 H_0^4}\; T^2(k)\, D_1^2(a)\, P_{\Phi_p}(\vec{k})$$

Since the perturbations are stochastic, this is the quantity of observational interest.

Now recall that $k^3 P_{\Phi_p}(\vec{k}) \propto k^{n_s-1}$ with n_s close to one. On scales that enter after matter-radiation equality, we should expect $P \propto k^{n_s}$, before the spectrum turns over and decays on smaller scales due to the damping in the radiation phase that is described by $T(k)$.

We can indeed observe this qualitative behaviour in the Figure. It also exhibits one of the many reasons why the turnover is of



interest: A Universe with a cosmological constant [ΛCDM] contains less CDM than one without a cosmological (sCDM). Matter-radiation equality occurs therefore later, on larger scales, for ΛCDM than for sCDM, which

is what the Figure exhibits.
We should also note that on very small scales, the linear approximation breaks down. This is indicated by the vertical line. Defining

$$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2}$$

it turns out that non-linearity roughly occurs when $\Delta(k_{nl}) \simeq 1$, where typically, $k_{nl} \simeq 0{,}2\, h\, Mpc^{-1}$.
Now, that we have already looked ahead into the qualitative features of the solutions, we also want to derive these.

5.2 Governing Equations

We have seen that long enough before couplings, the higher multipoles of the temperature perturbation are suppressed. Toward decoupling and after, all multipoles should be included (what is of importance for the CMB). However, at these times, the Universe is already matter dominated, such that we do not need to take account of the higher multipols in order to calculate the density perturbation of CDM, that is relevant for the LSS.
We truncate the multipoles beyond the dipole from the Boltzmann equations of Chapter 2. We also recall from Section 3.2 that in absence of quadrupols, we may set $\Phi = -\Psi$ and the definition

$$\Theta_\ell = \frac{1}{(-i)^\ell} \int_{-1}^{1} \frac{d\mu}{2}\, P_\ell(\mu)\, \Theta(\mu)$$

where $P_0(\mu) = 1$ and $P_1(\mu) = \mu$

We perform the multipole expansion by taking moments of the equation

$$\Theta_r' + ik\mu\,\Theta_r = -\Phi' - ik\mu\Psi = -\Phi' + ik\mu\Phi$$

$\Longrightarrow$

$$\Theta_{r0}' + k\,\Theta_{r1} = -\Phi'$$

$$\Theta_{r1}' - \frac{k}{2}\int_{-1}^{1}\mu^2 d\mu\,\Theta(\mu) = -\frac{k}{2}\int_{-1}^{1}\mu^2 d\mu\,\Phi \Longrightarrow \Theta_{r1}' - \frac{k}{3}\Theta_{r0} = -\frac{k}{3}\Phi$$

The subscript r indicates, that these equations hold for the sum of the temperature perturbation Θ and the neutrino perturbation $\mathcal{N}$, i.e. $\Theta_r = \Theta + \mathcal{N}$.

We also use the equations for the CDM perturbations

$$\delta' + ikv = -3\Phi'$$

$$v' + \frac{a'}{a}v = ik\Phi$$

Due to the various components of the equations for the metric perturbation, we may choose which ones to use. The following two forms are useful in different contexts:

$$3\frac{a'}{a}\left(\Phi' + \frac{a'}{a}\Phi\right) + k^2\Phi = 4\pi a^2 G\left[\rho_m\delta_m + 4\rho_r\Theta_r\right] \quad \text{(see Chap. 3.5)}$$

$$k^2\Phi = 4\pi G a^2\left[\rho_{DM}\delta + 4\rho_r\Theta_{r0} + \frac{3aH}{k}\left(i\underbrace{\rho_m v_m}_{=\rho_{DM}v + \rho_b v_b} + 4\rho_r\Theta_{r1}\right)\right]$$

The latter equation is algebraic (no derivatives) and can be obtained as a combination of the time-time component with the time-space components of the Einstein equations.

These five coupled equations (choose one of the metric equations) can straightforwardly be solved numerically.

To gain analytic understanding of the solutions, we can consider limiting cases. We can suitably divide the space of solutions as indicated in the figure. The solid curve is the comoving horizon (i.e. η for $a_0=1$). Note that at matter-radiation equality, $\eta \approx 30\,h^{-1}$ Mpc. A certain scale k is superhorizon at early times ($k\eta \ll 1$)

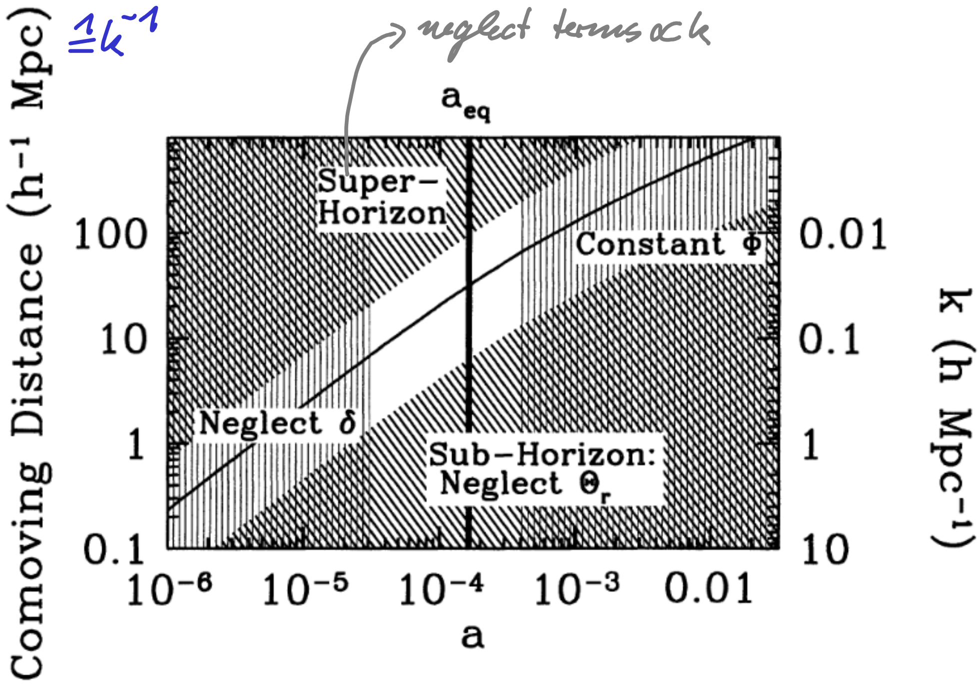


Figure 7.5. Physics of the transfer function. Hatched regions show where analytic expressions exist. The gaps in the center show that no analytic solutions exist to capture the full evolution of intermediate scale modes. The curve monotonically increasing from bottom left to top right is the comoving horizon.

and subsequently becomes subhorizon ($k\eta \gg 1$). Other useful approximations are indicated in the Figure and are explained in the following. We also see that a matching of the solutions is possible everywhere except for the interesting region $0{,}01\,h\,\mathrm{Mpc}^{-1} \lesssim k \lesssim 0{,}5\,h\,\mathrm{Mpc}^{-1}$. One may however spline the respective solutions together to obtain a reasonable fit.

5.3 Large Scales

On superhorizon scales, we may drop the terms that explicitly depend on k, such that we have to deal with

$$\Theta_{r0}' = -\Phi'$$

$$\delta' = -3\Phi'$$

$$3\frac{a'}{a}\left(\Phi' + \frac{a'}{a}\Phi\right) = 4\pi a^2 G\left[\rho_{DM}\delta + 4\rho_r \Theta_r\right]$$

We have neglected here baryons and we see that the equations for the fluid velocity and for the dipole decouple.

The first two equations tell us that $\Theta_{r0} - \frac{\delta}{3} = \text{const.}$ (as we already found in Chapter 4) and the constant is zero for adiabatic perturbations. We introduce

$$y = \frac{a}{a_{eq}} = \frac{\rho_{DM}}{\rho_r}$$

and substitute above relation into the metric equation, such that we obtain

$$3\frac{a'}{a}\left(\Phi' + \frac{a'}{a}\Phi\right) = 4\pi a^2 G \rho_{DM}\, \delta \left(1 + \frac{4}{3y}\right)$$

Now, transform variables to

$$\frac{d}{d\eta} = \frac{dy}{d\eta}\frac{d}{dy} = \frac{a^2}{a_{eq}}\frac{a'}{a^2}\frac{d}{dy} = aHy\frac{d}{dy} \qquad \frac{a'}{a^2} = H$$

We note that

$$\frac{8\pi G}{3}\rho_{DM} = \frac{8\pi G}{3}\rho\frac{\rho_{DM}}{\rho_{DM}+\rho_r} = H^2\frac{y}{y+1}$$

The metric equation then turns into (multiply by $\frac{1}{3a^2H^2}$

$$y\frac{d\Phi}{dy} + \Phi = \frac{y}{2(y+1)}\delta\left(1 + \frac{4}{3y}\right) = \frac{3y+4}{6(y+1)}\delta$$

Using the Boltzmann equation, we obtain

$$\frac{d\delta}{dy} = -3\frac{d\Phi}{dy} = \frac{d}{dy}\left[\frac{6(y+1)}{3y+4}\left(y\frac{d\Phi}{dy}\right) + \Phi\right]$$

This equation can be rearranged as

$$\frac{d^2\Phi}{dy^2} + \frac{21y^2+54y+32}{2y(y+1)(3y+4)}\frac{d\Phi}{dy} + \frac{\Phi}{y(y+1)(3y+4)} = 0$$

It is possible to find an analytic solution to this equation. We thus obtain for the potential on super-horizon scales:

$$\Phi = \Phi(0) \frac{1}{10y^3} \left(16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16\right)$$

For small y, we can convince ourselves that $\Phi \to \Phi(0)$ whereas for large y, we immediately see that $\Phi \to \frac{9}{10}\Phi(0)$. This suppression of Φ as it passes through matter-radiation equality is what we have noted earlier.

In the following figure, the analytical and the numerical results are compared:

We also indicate the point of horizon crossing with a red star. Beyond this, the approximation $k\eta << 1$ is no longer valid, but we would like to

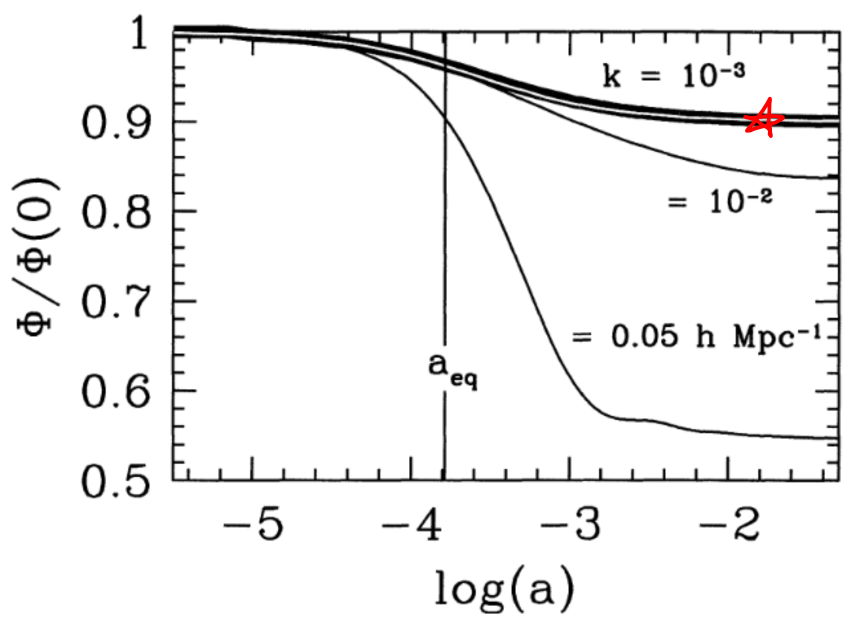


Figure 7.6. Super-horizon evolution of the potential in a CDM model with no baryons, $h = 0.5$ and $\Omega_m = 1$. Thick solid line shows the analytic result of Eq. (7.32), valid only on large scales. White curve within is for the mode $k = 0.001\,h$ Mpc^{-1}. Two other smaller scale modes are shown.

demonstrate the apparent feature that the potential remains constant through horizon crossing. Now we are deep in the matter-dominated regime, such that we can neglect the radiation perturbation. That leaves us with

$$\delta' + ikv = -3\Phi'$$

$$v' + \frac{a'}{a}v = ik\Phi$$

$$k^2 \Phi = 4\pi G a^2 \left[\rho_{DM} \delta + \frac{3aH}{k} i \rho v \right]$$

Note here that since $\rho_{DM} \propto a^{-3}$, a constant Φ implies that $\delta \propto a$.

We then need to show that the following set of admits a solution with constant Φ:

$$\delta' + ikv = 0, \quad v' + aHv = ik\Phi, \quad k^2\Phi = \frac{3}{2} a^2 H^2 \left(\delta + i \frac{3aHv}{k} \right).$$

During matter domination, $H \propto a^{-\frac{3}{2}}$, such that

$$\frac{a'}{a^2} \propto a^{-\frac{3}{2}} \Rightarrow a' \propto a^{\frac{1}{2}} \Rightarrow \frac{da}{\sqrt{a}} \propto d\eta \Longrightarrow \sqrt{a} \propto \eta \Longrightarrow a = \alpha \eta^2$$

$$H = \frac{2\alpha\eta}{\alpha^2 \eta^4} = \frac{2}{\alpha\eta^3} \Longrightarrow \frac{d}{d\eta}(aH) = \frac{d}{d\eta} \frac{2}{\eta} = -\frac{2}{\eta^2} = -\frac{1}{2} a^2 H^2$$

Then, we use the third equation to replace in the first equation δ with Φ and v:

$$\frac{2k^2\Phi'}{3a^2H^2} + \frac{2k^2\Phi}{3aH} - i\frac{3aHv'}{k} + i\frac{3a^2H^2v}{2k} + ikv = 0$$

Next, we eliminate (using the second equation) v':

$$\frac{2k^2\Phi'}{3a^2H^2} + \left(\frac{iv}{k} + \frac{2\Phi}{3aH} \right) \left(\frac{9a^2H^2}{2} + k^2 \right) = 0$$

We differentiate this equation once more and drop all terms $\propto \Phi'$ and Φ''. If the remaining terms can be solved for $\Phi = \text{const}$, the conservation of Φ is demonstrated. Note also that $\frac{d}{d\eta}\frac{1}{aH} = \frac{d}{d\eta}\frac{\eta}{2} = \frac{1}{2}$. We find

$$\left(i\frac{v'}{k} + \frac{\Phi}{3} \right) \left(\frac{9a^2H^2}{2} + k^2 \right) + \left(\frac{iv}{k} + \frac{2\Phi}{3aH} \right) \frac{d}{d\eta} \frac{9a^2H^2}{2}$$

$$= -\left(i\frac{aHv}{3} + \frac{2}{3}\Phi \right) (9a^2H^2 + k^2)$$

We have eliminated here v' terms by using the velocity equation above. The first factor on the right $\propto \Phi'$

by virtue of the previous intermediate equation. During the matter-dominated era, Φ is therefore conserved. In our Universe, Φ starts decaying again with the onset of Λ-domination.

5.4 Small Scales

We now consider the modes that enter the horizon before matter-radiation equality. We divide the analysis into the horizon-crossing during radiation domination and the subsequent evolution of the subhorizon modes through matter-radiation equality.

Horizon Crossing

During radiation domination, we also need to track the evolution of the matter perturbations, but we may neglect their backreaction on the metric and the radiation perturbations. We therefore need to solve a system of equations that is coupled according to the following scheme:

$$\Theta_{r0,1} \longleftrightarrow \Phi \longrightarrow \delta, v$$

We choose the algebraic equation for Φ and drop the matter source term

$$k^2 \Phi = 4\pi G a^2 \left(4\rho_r \Theta_{r0} + \frac{3aH}{k} 4\rho_r \Theta_{r1} \right)$$

$$\Longrightarrow \Phi = \frac{6a^2H^2}{k^2} \left(\Theta_{r0} + \frac{3aH}{k} \Theta_{r1} \right)$$

In the radiation era, $H \propto a^{-2}$, such that

$$\frac{a'}{a^2} \propto a^{-2} \Rightarrow a' = \text{const.} \Rightarrow a \propto \eta \Rightarrow aH = \frac{a'}{a} = \frac{1}{\eta}$$

We make use of this relation as well as of the Einstein equation above to eliminate Θ_{r0} from

$$\Theta_{r0}' + k\,\Theta_{r1} = -\Phi' \quad \text{and} \quad \Theta_{r1}' - \frac{k}{3}\Theta_{r0} = -\frac{k}{3}\Phi$$

We obtain

$$-\frac{3}{k\eta}\Theta_{r1}' + k\,\Theta_{r1}\left(1+\frac{3}{k^2\eta^2}\right) = -\Phi'\left(1+\frac{k^2\eta^2}{6}\right) - \Phi\,\frac{k^2\eta}{3}$$

and

$$\Theta_{r1}' + \frac{1}{\eta}\Theta_{r1} = -\frac{k}{3}\Phi\left(1-\frac{k^2\eta^2}{6}\right)$$

Now, use the second equation to eliminate Θ_{r1}' from the first, such that

$$\Phi' + \frac{1}{\eta}\Phi = -\frac{6}{k\eta^2}\Theta_{r1}$$

Differentiating once more yields

$$\Phi'' + \frac{1}{\eta}\Phi' - \frac{1}{\eta^2}\Phi = \frac{12}{k\eta^3}\Theta_{r1} - \frac{6}{k\eta^2}\Theta_{r1}'$$

$$= -\frac{3}{\eta}\Phi' - \frac{3}{\eta^2}\Phi + \frac{2}{\eta^2}\Phi\left(1-\frac{k^2\eta^2}{6}\right)$$

$\Longrightarrow$

$$\Phi'' + \frac{4}{\eta}\Phi' + \frac{k^2}{3}\Phi = 0$$

We solve this equation imposing the initial condition that Φ is constant on superhorizon scales. When we define $u = \Phi\eta$, we obtain

$$u'' + \frac{2}{\eta}u' + \left(\frac{k^2}{3} - \frac{2}{\eta^2}\right)u = 0$$

We recognise this as a spherical Bessel equation of order one with the solutions $j_1\left(\frac{k\eta}{3}\right)$ and $n_1\left(\frac{k\eta}{3}\right)$, where we can discard the Neumann-solution, because it diverges for $\eta \to 0$. The function j_1 can be expressed in terms of trigonometric functions, and we obtain

$$\Phi = 3\,\Phi_p \frac{\sin\frac{k\eta}{\sqrt{3}} - \frac{k\eta}{\sqrt{3}}\cos\frac{k\eta}{\sqrt{3}}}{\left(\frac{k\eta}{\sqrt{3}}\right)^3}$$

wher Φ_p is the primordial (superhorizon) value of Φ.

Typical solutions for modes that enter before equality are shown in the Figure. This confirms our anticipation, that Φ decays during the radiation epoch.

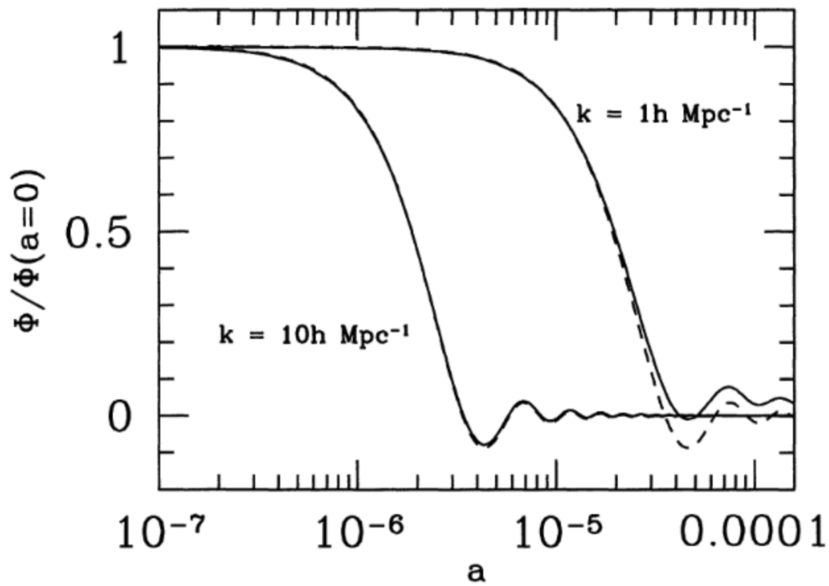


Figure 7.8. Evolution of the potential in the radiation-dominated era. For two small scale modes which enter the horizon well before equality, the exact (solid curve) solution is shown along with the approximate analytic solution (dashed curve) of Eq. (7.46).

Now we can go for the CDM perturbation δ in the background of given Φ (which is a valid approximation during radiation domination). We therefore take the equations for the matter perturbation

$$\delta' + ikv = -3\Phi' \implies v = \frac{i}{k}(3\Phi' + \delta')$$

$$v' + \frac{a'}{a}v = ik\Phi \implies v' = -\frac{i}{k}\frac{a'}{a}(3\Phi' + \delta') + ik\Phi$$

$\implies$

$$\delta'' + \frac{a'}{a}(3\Phi' + \delta') - k^2\Phi = -3\Phi''$$

which we express as

$$\delta'' + \frac{1}{\eta}\delta' = S(k,\eta) \quad \text{where the source term is}$$

$$S(k,\eta) = -3\Phi'' - \frac{3}{\eta}\Phi' + k^2\Phi$$

The solutions to the homogeneous equation are $\delta = \text{const.}$ and $\delta = \log a$. We construct the inhomogeneous solution with the help of a Green function:

$$G(\eta', \eta) = -\vartheta(\eta-\eta')\left[\eta' \log(k\eta') - \eta' \log(k\eta)\right]$$

$$\partial_\eta G(\eta', \eta) = \vartheta(\eta-\eta') \frac{\eta'}{\eta}$$

$$\partial^2_{\eta'} G(\eta', \eta) = \delta(\eta-\eta') - \vartheta(\eta-\eta') \frac{\eta'}{\eta^2}$$

$$\Longrightarrow \left[\partial^2_\eta + \frac{1}{\eta}\partial_\eta\right] G(\eta', \eta) = \delta(\eta-\eta')$$

$\Longrightarrow$

$$\delta(k, \eta) = C_1 + C_2 \log \eta - \int_0^\eta d\eta' S(k, \eta')\eta'\left[\log(k\eta') - \log(k\eta)\right]$$

Since at very early times, all scales become superhorizon and should be constant, we must set $C_2 = 0$ and $C_1 = 3\Theta_0\Big|_{\eta\to 0} = \frac{3}{2}\Phi_P$ (cf. Chapter 4)

Now, the term S decays along with Φ after horizon entry. For $k\eta \gg 1$, integral term involving $\log(k\eta')$ will therefore yield a constant contribution to δ, while the other term will yield a logarithmically growing contribution $\propto \log(k\eta)$. We can therefore make the ansatz

$$\delta(k, \eta) = A\Phi_P \log(Bk\eta)$$

where

$$A\Phi_P \log B = \frac{3}{2}\Phi_P - \int_0^\infty d\eta' S(k, \eta')\eta' \log(k, \eta')$$

$$A\Phi_P = \int_0^\infty d\eta' S(k, \eta')\eta'$$

Using the solutions for Φ, one then finds that $A = 9{,}0$ and $B = 0{,}62$.

The numerical solution confirms the logarithmic growth after horizon entry. Recall for comparison that during matter-domination, $\delta \propto a$. This difference is due to the fact that the radiation pressure antagonises the growth.

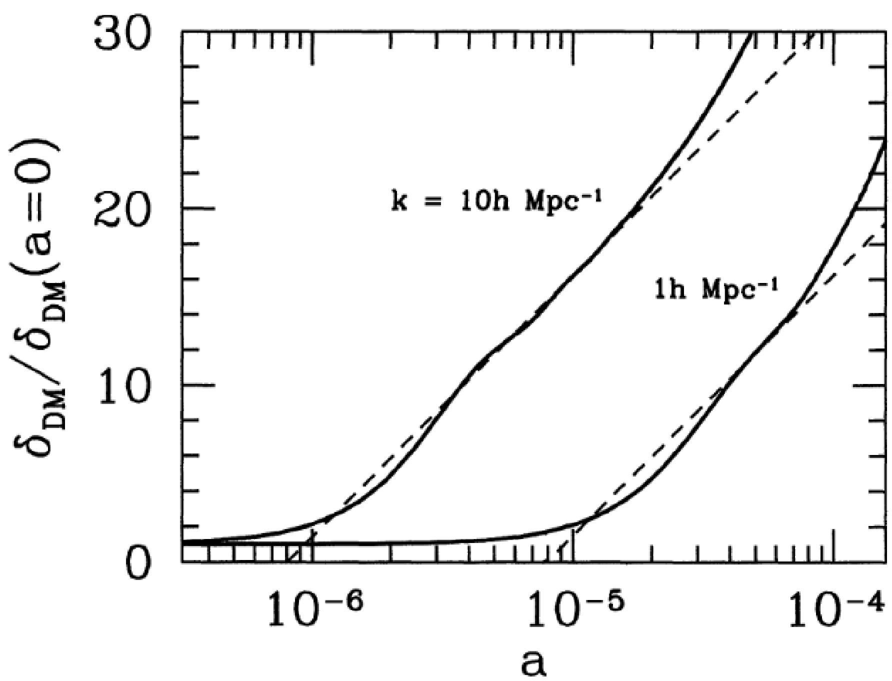


Figure 7.9. Matter perturbations in the radiation-dominated era. The two scales shown here both enter the horizon in the radiation era and lock onto the logarithmically growing mode after some oscillations. Heavy solid curves are the exact solutions, light dashed curves the logarithmic mode of Eq. (7.51). The perturbations have been artificially normalized by their values at early times: inflation actually predicts a larger initial amplitude (by a factor of $10^{3/2}$) for the larger scale mode.

Sub-Horizon Evolution

The growth of the matter perturbations inside the horizon implies that these dominate over the radiation perturbations already at some point before the equality. The growth of the radiation perturbation is inhibited by the pressure, as we will see in the next Chapter. Now, we therefore solve for the common evolution of CDM and gravitational potentials, neglecting radiation, as it is governed by the equations

(recall that $y = \frac{a}{a_{eq}} = \frac{\rho_{DM}}{\rho_r}$ and $\frac{d}{d\eta} = aHy\frac{d}{dy}$)

$$\frac{d}{dy}\delta + \frac{ikv}{aHy} = -3\frac{d}{dy}\Phi \qquad (1+\frac{1}{y})\varrho_{DM} = (\varrho_{DM}+\varrho_r)$$

$$\frac{d}{dy}v + \frac{v}{y} = \frac{ik\Phi}{aHy}$$

$$k^2\Phi = \frac{3ya^2H^2}{2(1+y)}\delta$$ where we use $\varrho_{DM}+\varrho_r = \varrho_{DM}(1+y)$

We have neglected here terms $\propto \frac{aHv}{k}$, because

$\frac{aH}{k} = \frac{a'}{ak} = \frac{1}{k\eta} \ll 1$ inside the horizon.

We differentiate the first equation and use the second one to obtain

$$\frac{d^2\delta}{dy^2} + \frac{ik}{aHy}\left(-\frac{v}{y} + \frac{ik\Phi}{aHy}\right) - \frac{ikv}{2aHy(1+y)}$$

$$\frac{d^2}{dy^2}\delta + \frac{ikv(2+3y)}{2aHy^2(1+y)} - \frac{k^2\Phi}{a^2H^2y^2} = -3\frac{d^2}{dy^2}\Phi$$

where we have used that

$$H^2 = \frac{8\pi}{3m_{Pl}^2}(\varrho_{DM}+\varrho_r) = \frac{8\pi}{3m_{Pl}^2}\varrho_{DM}\frac{1+y}{y} = \alpha\, a^{-3}\frac{1+y}{y} = \alpha\, a_{eq}^{-3}\frac{1+y}{y^4}$$

$$aHy = a_{eq}y^2\sqrt{\alpha}\, a_{eq}^{-\frac{3}{2}}\frac{\sqrt{1+y}}{y^2}$$

$$\frac{d}{dy}\frac{1}{aHy} = \frac{\sqrt{a_{eq}}}{\sqrt{\alpha}}\frac{d}{dy}(1+y)^{-\frac{1}{2}} = -\frac{1}{2}\frac{\sqrt{a_{eq}}}{\sqrt{\alpha}}\frac{1}{\sqrt{1+y}}\frac{1}{1+y} = -\frac{1}{aHy}\frac{1}{1+y}$$

We can use the Einstein equation, to replace

$$\frac{k^2\Phi}{a^2H^2y^2} = \frac{3\delta}{2y(y+1)}$$

and the density perturbation equation to write

$$\frac{ikv}{aHy} = -\frac{d}{dy}\delta$$

where we neglect $\frac{d}{dy}\Phi$ because the potential suffers decay inside the horizon during radiation domination, while δ grows. Besides, we neglect the $\frac{d^2}{dy^2}\Phi$ term, because the $\frac{k^2}{a^2H^2}\Phi$ term is much larger in comparison, because $k\eta \gg 1$.

In summary, we obtain

$$\frac{d^2}{dy^2}\delta + \frac{2+3y}{2y(y+1)}\frac{d}{dy} - \frac{3\delta}{2y(y+1)} = 0$$

which is known as the Meszaros equation. It describes the subhorizon evolution of CDM perturbations once the radiation perturbations have become negligible.

The solutions to this equation can be expressed as a growing mode

$$D_1(y) = y + \frac{2}{3}$$

and a decaying mode

$$D_2(y) = D_1(y)\log\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1} - 2\sqrt{1+y}$$

such that

$$\delta(k,y) = C_1 D_1(y) + C_2 D_2(y)$$

These solution should be matched to the logarithmically growing solution that we have found before and that is valid before matter-radiation equality. (Notice that the validity of both solutions overlaps!)

We define y_H as the value of y at horizon entry of a certain mode and y_m defines the matching point.

The matching conditions for the zeroth and first derivatives read

$$A\,\Phi_p \log\left(B\frac{y_m}{y_H}\right) = C_1 D_1(y_m) + C_2 D_2(y_m)$$

$$\frac{A\,\Phi_P}{y_m} = C_1 \frac{d}{dy_m} D_1(y_m) + C_2 \frac{d}{dy_m} D_2(y_m)$$

To state more quantitatively when this procedure is valid, we should require that $y_H \ll y_m \ll 1$, i.e. mode entry and matching occur prior to matter-radiation equality.

To obtain the power spectrum, we next need to determine the transfer function. We therefore solve for the coefficient of the growing solution

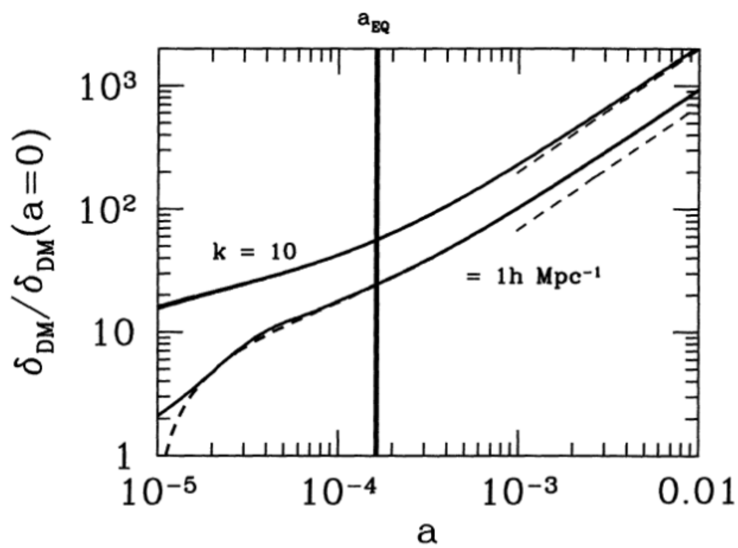


Figure 7.10. Evolution of small-scale, sub-horizon, dark matter perturbations. Solid curves are exact solutions; dashed curves (almost imperceptible because the goodness of fit in the $10\,h$ Mpc^{-1} case) the Meszaros solution with coefficients given by the matching condition, Eq. (7.64). The dashed straight lines at $a > 10^{-3}$ are the asymptotic solution of Eq. (7.67).

$$C_1 = \frac{A \log\left(B \frac{y_m}{y_H}\right) \frac{d}{dy_m} D_2(y_m) - D_2(y_m) \frac{A}{y_m}}{D_1(y_m) \frac{d}{dy_m} D_2(y_m) - D_2(y_m) \frac{d}{dy_m} D_1(y_m)} \, \Phi_P$$

We find for the denominator

$$D_1(y_m) \frac{d}{dy_m} D_2(y_m) - D_2(y_m) \frac{d}{dy_m} D_1(y_m) = -\frac{4}{9} \frac{1}{y_m \sqrt{1+y_m}} \underset{y_m \ll 1}{\simeq} -\frac{4}{9} y_m$$

and

$$D_2(y_m) \underset{y_m \ll 1}{\simeq} \frac{2}{3} \log \frac{4}{y_m} - 2, \quad \frac{d}{dy_m} D_2(y_m) \underset{y_m \ll 1}{\simeq} -\frac{2}{3} y_m$$

It follows

$$C_1 \simeq -\frac{9}{4} A\, \Phi_P \left[-\frac{2}{3} \log \frac{4B}{y_H} + 2 \right]$$

and

$$\delta(\vec{k},a) = \frac{3A\Phi_P(\vec{k})}{2} D_1(a) \log \frac{4Be^{-3}a_{eq}}{a_H}$$

Now, we define the scale factors a_H, a_{eq} through the instance, when the physical momentum scale $\frac{k}{a}$ coincides with the Hubble scale, i.e.

$$a_H H(a_H) = k \quad \text{and} \quad a_{eq} H(a_{eq}) = k_{eq}$$

Then, we note that

$$e_r^{eq} = e_m^{eq} = \left(\frac{a_H}{a_{eq}}\right)^4 e_r^H$$

such that

$$H(a_H) \simeq \sqrt{\frac{8\pi}{3}} \frac{\sqrt{e_r^H}}{m_{Pl}} \quad \text{and} \quad H(a_{eq}) \simeq \sqrt{\frac{8\pi}{3}} \frac{1}{m_{Pl}} \sqrt{2} \left(\frac{a_H}{a_{eq}}\right)^2 \sqrt{e_r^H}$$

$$\Longrightarrow \frac{H(a_{eq})}{H(a_H)} = \sqrt{2}\left(\frac{a_H}{a_{eq}}\right)^2 \Longrightarrow \frac{a_H}{a_{eq}} = \frac{k}{k_{eq}} \frac{H(a_{eq})}{H(a_H)} = \sqrt{2} \frac{k}{k_{eq}} \left(\frac{a_H}{a_{eq}}\right)^2$$

We can therefore replace $\frac{a_{eq}}{a_H} = \sqrt{2}\frac{k}{k_{eq}}$, such that we obtain the following form for the transfer function on small scales

$$T(k) = \frac{5A\Omega_m H_0^2}{2k^2 a_{eq}} \log \frac{4Be^{-3}\sqrt{2}k}{k_{eq}} \quad \text{for} \quad k \gg k_{eq}$$

As for the prefactor, we replace

$$k_{eq} = a_{eq} H(a_{eq}) = a_{eq} \underbrace{\sqrt{2}}_{\text{matter and radiation}} \underbrace{\left(\frac{a_0}{a_{eq}}\right)^{\frac{3}{2}}}_{\text{matter scaling}} H_0 = \frac{1}{\sqrt{a_{eq}}}\sqrt{2} H_0$$

(assuming CDM domination at late times — present Universe is however dominated by Λ !)

$$\frac{5A\Omega_m H_0^2}{2k^2 a_{eq}} = \frac{45 H_0^2 k_{eq}^2}{4k^2 H_0} \simeq \frac{12 k_{eq}^2}{k^2}$$

Similarly, we plug in the numbers in the argument of the logarithm to obtain the approximate form

$$T(k) \simeq \frac{12\, k_{eq}^2}{k^2} \log \frac{k}{8 k_{eq}}$$

This can be splined together with the large scales, for which $T(k) \equiv 1$ by definition.

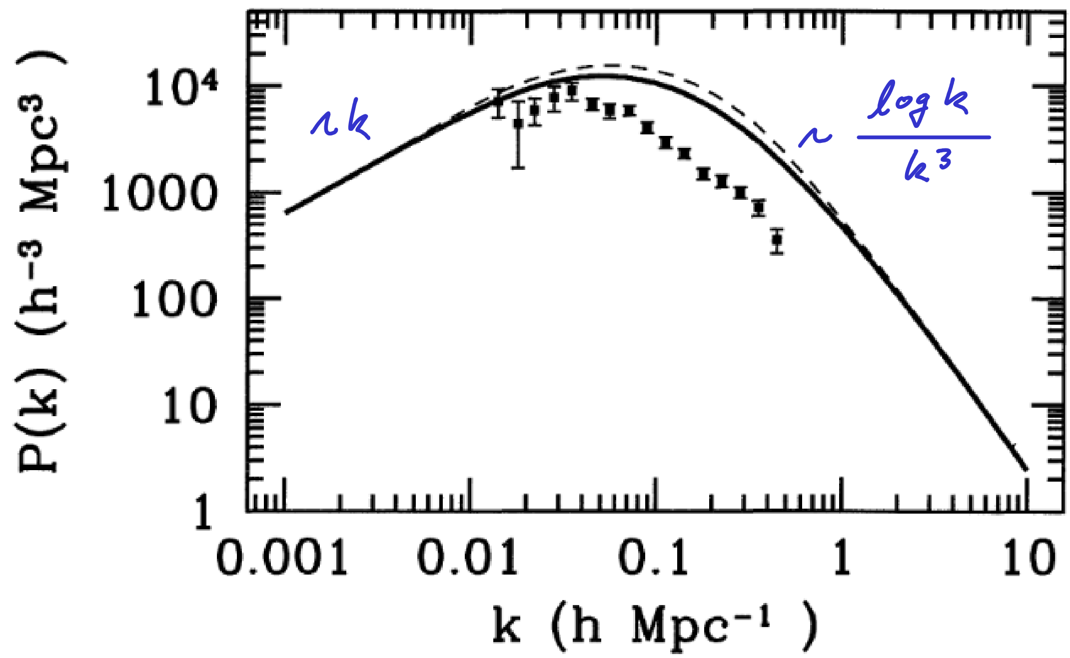


Figure 7.11. The power spectrum in a standard CDM model with a Harrison–Zel'dovich–Peebles spectrum. The thick solid curve uses the BBKS transfer function; the dashed curve interpolates between the analytic transfer function on large scales (equal to 1) and small scales (Eq. (7.68)). The data points are a compilation (and interpretation) by Peacock and Dodds (1994).

We have indicated the approximate behaviour for $n_s \simeq 0$.

* On large scales, Φ is constant, but δ grows after horizon entry in the matter dominated era.
* On small scales, the radiation pressure leads to a decay of Φ for modes that enter during radiation domination. As a consequence, δ is suppressed as well.

5.5 Growth Function

Now that the shape of $P(k)$ is derived, we yet need to obtain the amplitude. The Meszaros equation describes the evolution in a background that consists of matter and radiation. We want to take account of a cosmological constant contribution, that should inhibit the growth at late times.

For this purpose, we consider once more

$$\frac{d}{dy}\delta + \frac{ikv}{aHy} = -3\frac{d}{dy}\Phi \quad \text{and} \quad \frac{d}{dy}v + \frac{v}{y} = \frac{ik\Phi}{aHy}$$

$\Longrightarrow$

$$\frac{d^2}{dy^2}\delta + ikv\frac{d}{dy}\frac{1}{aHy} - \frac{ikv}{aHy^2} - \frac{k^2}{a^2H^2y^2}\Phi = -3\underbrace{\frac{d}{dy}\Phi}_{\text{neglect: } k\eta >> 1}$$

We then neglect radiation in the Poisson equation

$$k^2\Phi = 4\pi G a^2 \rho_{DM}\delta = \frac{3}{2}H_0^2\Omega_m\frac{\delta}{a}$$

such that

$$\frac{d^2}{dy^2}\delta + ikv\left(\frac{d}{dy}\frac{1}{aHy} - \frac{1}{aHy^2}\right) = \frac{3\Omega_m H_0^2}{2y^2a^3H^2}\delta$$

Again, we use the approximation $\frac{ikv}{aHy} = -\frac{d}{dy}\delta$ (neglect Φ compared to δ) and we use $\frac{d}{dy} = a_{eq}\frac{d}{da}$ such that

$$\frac{d}{dy}\frac{1}{aHy} = a_{eq}\frac{d}{da}\frac{1}{\frac{a^2}{a_{eq}}\frac{\dot a}{a}} = a_{eq}^2\frac{d}{da}\frac{1}{a\dot a} = a_{eq}^2\left(-\frac{1}{a^2\dot a} - \frac{\frac{d}{da}\dot a}{a\dot a^2}\right)$$

$$aHy = \frac{a^2}{a_{eq}}H$$

$$\frac{1}{a^2H}a_{eq}^2\frac{d}{da}\log\left(\frac{\dot a}{a}\right) = \frac{1}{a\dot a}a_{eq}^2\frac{a}{\dot a}\left(\frac{1}{a}\frac{d}{da}\dot a - \frac{\dot a}{a^2}\right) = a_{eq}^2\left(-\frac{1}{a^2\dot a} + \frac{\frac{d}{da}\dot a}{a\dot a^2}\right)$$

Thus,

$$a_{eq}^2\frac{d^2}{da^2}\delta + aHy\left(\frac{a_{eq}^2}{a^2H}\frac{d}{da}\log(H) + 2\frac{a_{eq}^2}{a^2\dot a} + \frac{1}{aHy^2}\right)a_{eq}\frac{d}{da}\delta$$

$$- a_{eq}^2\frac{3\Omega_m H_0^2}{2a^5H^2}\delta = 0$$

$\Longrightarrow$

$$\frac{d^2}{da^2}\delta + \left(\frac{d}{da}\log H + \frac{3}{a}\right) - \frac{3\Omega_m H_0^2}{2a^5 H^2}\delta = 0$$

Note that in this large y limit, all reference to a_{eq} disappears. The growing solution to this equation gives rise to the growth factor (the normalisation is fixed by the requirement that at early times, $D_1(a) = a$)

$$D_1(a) = \frac{5\Omega_m}{2}\frac{H(a)}{H_0}\int_0^a \frac{da'}{\left(a'\frac{H(a')}{H_0}\right)^3}$$

Exemplary solutions are presented in the Figure, from which we can observe that curvature or cosmological constant contributions suppress the growth at late times.

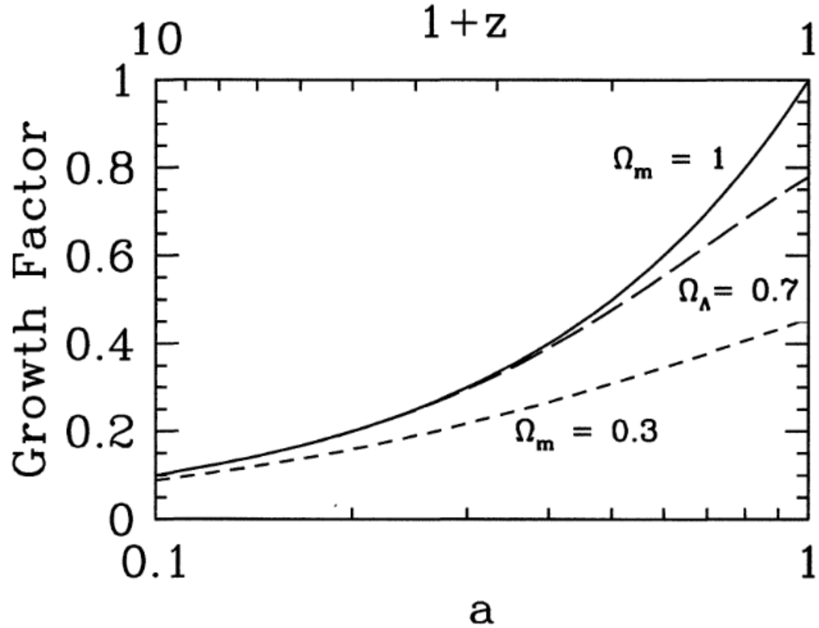


Figure 7.12. The growth factor in three cosmologies. Top two curves are for flat universes without and with a cosmological constant. Bottom curve is for an open universe.

5.6 Beyond CDM

Adding baryons, neutrinos and a cosmological constant modifies the analytic estimates worked out above. The spectra can be obtained straightforwardly from numerical solutions to the linearised perturbation equations

The Figure indicates that these corrections mostly lead to a suppression of power on small scales. The various different reasons for why this happens for the particular components are discussed in the following.

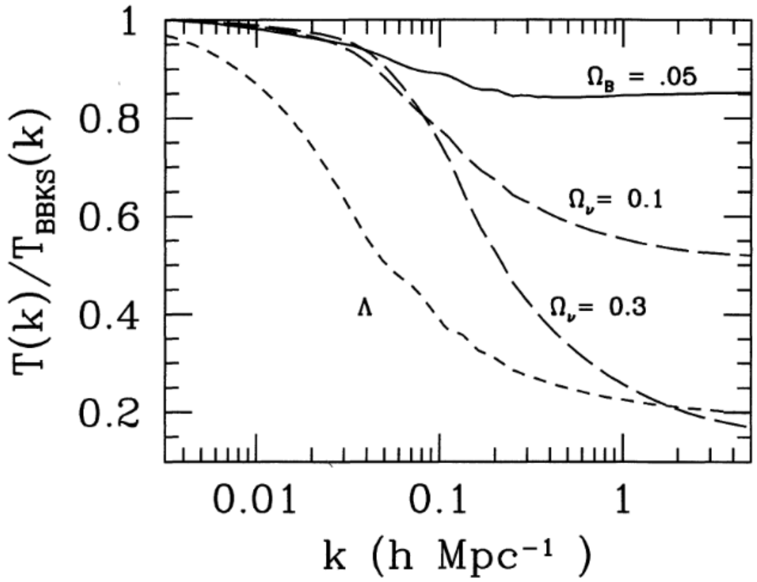


Figure 7.13. The ratio of the transfer function to the BBKS transfer function (Eq. (7.70)) which describes dark-matter-only (no baryons) perturbations. Top curve (and all other curves as well) has 5% baryons. Two middle curves show different values for a massive neutrino. Bottom curve has a cosmological constant $\Omega_\Lambda = 0.7$.

Massive Neutrinos

Neutrinos are decouple and move from high density to low density regions (free streaming). Perturbations on scales smaller than the free streaming scale are therefore suppressed. To estimate this scale, note that the average neutrino velocity is $\frac{T_\nu}{m_\nu}$ and compare this to the Hubble rate. Larger m_ν means more efficient suppression but also that only smaller scales are affected. Currently, the bound on $\sum_{\nu_i} m_{\nu_i} \simeq 0{,}5\,eV$, what implies that all scales $< k_{eq}^{-1}$ are suppressed — but since the masses are so small, the effect has not been noticed yet. The cosmological determination of neutrino masses is among the main objectives of current and future galaxy surveys.

Baryons

5% of the energy density of the present Universe are made up by baryons. Upon horizon entry but before decoupling, baryons tightly couple to radiation and suffer the same suppression. Only after decoupling, the baryons fall into the Dark Matter potentials. This leads therefore to a suppression of power.

In addition, we note the wiggles around the equality scale. We have seen that Φ has an oscillatory component during radiation domination (due to the pressure). Due to the tight coupling, these oscillations are also present in the baryons and they eventually leave their small imprint in the matter power spectrum.

Importantly, these <u>baryon acoustic oscillations</u> have now been confirmed in the galaxy power spectrum, while ten years ago, Dodelson was still placing doubts on this.

<u>Dark Energy</u>

Dark Energy

- shifts k_{eq}^{-1} to larger values (less matter compared to radiation at early times)
- as stated above, it suppresses the growth at late times.