4. Initial Conditions

Now that we have derived Einstein-Boltzmann equations, we want to derive solutions that describe the cosmological observations. For this purpose, the input of initial perturbation spectra is needed, which we motivate in this Chapter.

4.1 The Einstein-Boltzmann Equations at Early Times

At sufficiently early times, any k-mode of interest satisfies $k\eta \ll 1$. In that situation, we may perform simplifying approximations on the Boltzmann equation for radiation:

$$\Theta' + ik\mu\Theta = -\Phi' - ik\mu\Psi - \tau'\left[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

First, derivatives may be counted as $\mathcal{O}\left(\frac{1}{\eta}\right)$, such that we may drop terms that feature an explicit factor of k. Second, the velocities are due to the large Compton scattering rate $v_b = -3i\Theta_1$ (cf. the equation for the baryon velocity) and moreover $\Theta \to \Theta_0$, corresponding to the suppression of all multipoles beyond the monopole. We hence come up with the simple approximations (and accordingly for neutrinos):

$$\Theta_0' + \Phi' = 0 \qquad \mathcal{N}_0' + \Phi' = 0$$

Next, consider the Einstein equation

$$3\frac{a'}{a}\left(\Phi' - \frac{a'}{a}\Psi\right) + k^2\Phi = 4\pi a^2 G\left[\rho_m\delta_m + \rho_b\delta_b + 4\rho_r\Theta_{r}\right]$$

Here, we drop the term $\propto k^2$, and make use of the fact that at early times, radiation dominates over matter. During radiation domination, $a \propto \eta$ and so $\frac{a'}{a} = \frac{1}{\eta}$. Moreover, we use the Friedmann equation

$$\mathcal{H}^2 = \frac{a'^2}{a^4} = \frac{1}{a^2\eta^2} = \frac{8\pi G}{3}\rho$$ such that we obtain

$$\frac{\Phi'}{\eta} - \frac{\Psi}{\eta^2} = \frac{16\pi}{3} a^2 \rho \left(\frac{\rho_\gamma}{\rho}\Theta_0 + \frac{\rho_\nu}{\rho}\mathcal{N}_0 \right) = \frac{2}{\eta^2}\left(\frac{\rho_\gamma}{\rho}\Theta_0 + \frac{\rho_\nu}{\rho}\mathcal{N}_0 \right)$$

Now, we define the ratio of the neutrino density to the total energy density

$$f_\nu = \frac{\rho_\nu}{\rho_\gamma + \rho_\nu}$$

such that we can rewrite above equation as

$$\Phi'\eta - \Psi = 2\left[(1-f_\nu)\Theta_0 + f_\nu \mathcal{N}_0 \right]$$

Next, we differentiate and substitute our simplified Boltzmann equations from above, such that we obtain

$$\Phi''\eta + \Phi' - \Psi' = -2\Phi'$$

The second scalar metric equation tells us that $\Phi = -\Psi$, up to corrections from higher multipols. For photons, these are suppressed due to Compton scatterings, whereas for the neutrinos, there are small corrections that we ignore for the present purposes. We hence obtain

$$\Phi''\eta + 4\Phi' = 0.$$

We find solutions to this equation using the ansatz $\Phi \propto \eta^p$ that leads to $p(p-1) + 4p = p^2 + 3p = 0 \Longrightarrow p = 0, -3.$

The $p=-3$ mode decays rapidly, but if the $p=0$ mode is excited initially, it may lead to the observed cosmic perturbations. Substituting this mode into above equation, we find the relation

$$\Phi = 2\left[(1-f_\nu)\Theta_0 + f_\nu \mathcal{N}_0 \right]$$

Hence, the photon and neutrino perturbations are constant at early times as well. Moreover, since photons and neutrinos used to be coupled before neutrino decoupling, we may set here $\Theta_0(k,\eta_i) = \mathcal{N}_0(k,\eta_i) \Longrightarrow \Phi(k,\eta_i) = 2\,\Theta_0(k,\eta_i)$, where η_i is some early time when we specify the initial conditions.

As for CDM, combining $\Theta_0' + \Phi' = 0$ with $\delta' + \underbrace{ikv}_{\approx 0} = -3\Phi'$, we find that

$$\delta = 3\,\Theta_0 + \text{const.}$$

A corresponding equation holds for the baryon overdensity as well. When the constant term is zero, we call the primordial perturbation <u>adiabatic</u>, whereas we call the contributions from a non-vanishing constant term <u>isocurvature</u> (as these do not change the metric perturbations).

For adiabatic perturbations, the matter-to-radiation ratio is constant everywhere in space-time, since

$$\frac{n_{DM}}{n_\gamma} = \frac{n_{DM}^{(0)}}{n_\gamma^{(0)}}\,\frac{1+\delta}{1+3\Theta_0} \quad \text{and to linear order} \quad \frac{1+\delta}{1+3\Theta_0} = 1 + \delta - 3\,\Theta_0 = 1$$

Similarly, for adiabatic perturbations in baryons, $\delta_b = 3\,\Theta_0$.

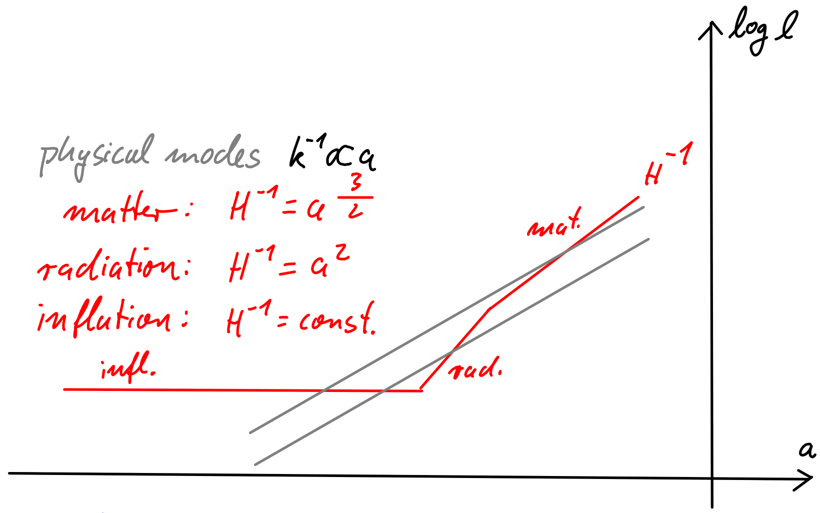
Isocurvature perturbations have not been observed so far, hence we will mostly focus on the purely adiabatic case here.

For completeness, we mention that one can show that for the dipoles and velocities, initial conditions are given by

$$\Theta_1 = \mathcal{N}_1 = \frac{i v_b}{3} = \frac{i v}{3} = -\frac{k\,\Phi}{6aH}$$

4.2 Primordial Spectra

Now, that we have simplified the problem by tying the initial perturbations to Φ, we need to come up with initial conditions for perturbations in Φ itself, as well as possible tensor modes.



The Universe observed in terms of galaxies and the CMB looks statistically isotropic, even on very large scales. Now, as the diagram exhibits, points separated by these scales have not been in causal contact when assuming radiation and matter dominated expansion only.

The most popular way to address this is to assume a period of accelerated expansion with an approximately constant Hubble rate at the earliest times, that is called inflation. One should note however that inflation is not the only proposal for a solution to this puzzle, nor is it proved observationally.

Now, inflation is of interest on its own and it deserves a detailled presentation at a later point. For now, we just state that quantum fluctuations in the scalar and tensorial perturbations naturally predict a scale-invariant power spectrum (since H is approximately constant).

For a given k-mode, we expect that $\langle \Phi(\vec{k}) \rangle = 0$.
The variance should however be given by

$$\langle \Phi(\vec{k}) \Phi(\vec{k}') \rangle = (2\pi)^3 P_\Phi(\vec{k}) \delta^3(\vec{k} - \vec{k}')$$

The δ-function implies that different k-modes are uncorrelated, in other words that the spectrum is Gaussian. Possible deviations from this property are referred to as non-Gaussianities.

For the purposes of inflation, it is useful to use the curvature perturbation ξ as the variable describing scalar perturbations. After inflation, it is related to Φ as $\xi = \frac{3}{2}\Phi$. The common parametrisation is to introduce dimensionless spectra

$$\mathcal{P}_X(k) = \frac{k^3}{2\pi^2} P_X(k)$$

For example, the scalar curvature perturbation is then given by

$$(2\pi)^3 \frac{2\pi^2}{k^3} \mathcal{P}_\xi \delta^3(\vec{k} - \vec{k}') = \langle \xi(\vec{k}) \xi(\vec{k}') \rangle = \frac{9}{4} \langle \Phi(\vec{k}) \Phi(\vec{k}') \rangle$$

For the tensors, one defines (assuming the same spectra for the + and × modes)

$$4 \langle h_+(\vec{k}) h_+(\vec{k}') \rangle = 4 \langle h_\times(\vec{k}) h_\times(\vec{k}') \rangle = (2\pi)^3 \frac{2\pi^2}{k^3} \mathcal{P}_h \delta^3(\vec{k} - \vec{k}')$$

If inflation proceeded at a constant expansion rate H, the dimensionless would be expected to be independent of k. However, when H is not exactly constant (as it is expected), there can be a deviation from scale-invariance, that is parametrised in terms of a scalar spectral index:

$$n_s - 1 = \frac{\partial \log \mathcal{P}_\zeta}{\partial k}$$

For example, the Planck collaboration parametrises

$$\mathcal{P}_\zeta(k) = A_s \left(\frac{k}{k_0} \right)^{n_s - 1}$$

where $k_0 = 0{,}05\ Mpc^{-1}$ is a so-called pivot scale.
Note that $n_s = 1$ corresponds to a scale-invariant spectrum.
Planck reports that

$$A_s = (2{,}196^{+0{,}051}_{-0{,}060}) * 10^{-9}$$

$$n_s = 0{,}9603 \pm 0{,}0073$$

Similarly, one may parametrise the tensor spectrum in terms of an amplitude and a spectral index.