3. Perturbed Einstein Equations

In the previous Chapter, we have derived Boltzmann equations for the radiation and matter components of the expanding Universe in the presence of metric perturbations described by the functions Φ and Ψ. Now, these metric perturbations are also dynamical, and we derive here the equations that govern these.

3.1 The Perturbed Ricci Tensor and Scalar

Our task is to express the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

in terms of Φ and Ψ, that enter the game through the perturbed metric

$$g_{00}(\vec{x}, t) = 1 + 2\Psi(\vec{x}, t)$$
$$g_{0i}(\vec{x}, t) = 0$$
$$g_{ij}(\vec{x}, t) = -a^2 \delta_{ij} (1 + 2\Phi(\vec{x}, t))$$

The tactics is straightforward: We first calculate the Christoffel Symbols and from these $R_{\mu\nu}$ and R.
We recall that the Christoffel symbols are given by

$$\Gamma^{\kappa}_{\mu\nu} = \frac{g^{\kappa\alpha}}{2} \left(\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\alpha}} \right)$$

and first go for $\kappa = 0$. The only non-zero component of $g^{0\alpha}$ is g^{00}, and to first order $g^{00} = 1 - 2\Psi$ and thus

$$\Gamma^{0}_{\mu\nu} = \frac{1-2\Psi}{2} \left(\frac{\partial g_{0\mu}}{\partial x^{\nu}} + \frac{\partial g_{0\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{0}} \right)$$

Now for the particular components. First consider $\mu = \nu = 0$, such that each term in the brackets is identical:

$$\frac{\partial g_{00}}{\partial x^0} = 2 \frac{\partial}{\partial x^0} \Psi$$

To first order, we then obtain

$$\Gamma^0_{00} = \frac{\partial}{\partial x^0} \Psi .$$

Next, we take one of μ, ν to be spatial, the other one zero.
We get contributions from $\frac{\partial g_{00}}{\partial x^i} = 2 \frac{\partial \Psi}{\partial x^i} \to 2 i k_i \Psi$, such that
$\Gamma^0_{0i} = \Gamma^0_{i0} = \frac{\partial \Psi}{\partial x^i} \longrightarrow i k^i \Psi$, where we also give the expressions in Fourier space.

When both lower indices are spatial, we obtain

$$\Gamma^0_{ij} = \frac{1-2\Psi}{2} \frac{\partial}{\partial t} \left[\delta_{ij}\, a^2 (1 + 2 \Phi(\vec{x}, t)) \right]$$

$$\stackrel{\text{1st order}}{=} \delta_{ij}\, a^2 \left[H + 2H(\Phi - \Psi) + \frac{\partial \Phi}{\partial t} \right]$$

Next, we calculate

$$\Gamma^i_{00} = - \frac{g^{ii}}{2} \frac{\partial g_{00}}{\partial x^i} = \frac{1}{a^2}(1 - 2\Phi) \frac{\partial \Psi}{\partial x^i} \longrightarrow \frac{1}{a^2} i k^i \Psi$$

Note here that the vector $\vec{k}$ lives in Euclidean space, such that we can freely exchange upper and lower indices. (The Fourier transform is always defined w.r.t. x^i).

$$\Gamma^i_{j0} = \Gamma^i_{0j} = \frac{g^{ii}}{2} \frac{\partial g_{ji}}{\partial x^0} = \frac{1}{2} \frac{1}{a^2} (1 - 2\Phi)\, \delta^i_j \frac{\partial}{\partial t} \left[a^2 (1 + 2\Phi) \right] = \delta^i_j \left(H + \frac{\partial \Phi}{\partial t} \right)$$

$$\Gamma^i_{jk} = \frac{g^{ii}}{2} \left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i} \right)$$

$$= (1 - 2\Phi) \left(\delta_{ki} \frac{\partial \Phi}{\partial x^j} + \delta^i_j \frac{\partial \Phi}{\partial x^k} - \delta_{jk} \frac{\partial \Phi}{\partial x^i} \right)$$

$$= i \Phi \left(\delta^i_j k_k + \delta^i_k k_j - \delta_{jk} k^i \right)$$

Now we have the necessary ingredients to derive an expression for the Ricci tensor

$$R_{\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}$$

and start with

$$R_{00} = \frac{\partial \Gamma^{\alpha}_{00}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}_{0\alpha}}{\partial x^{0}} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{00} - \Gamma^{\alpha}_{\beta 0}\Gamma^{\beta}_{0\alpha}$$

We immediately see that the $\alpha = 0$ contributions cancel and only need to care about spatial α. For these, we treat the four terms separately:

First term:

$$\frac{\partial \Gamma^{i}_{00}}{\partial x^{i}} = -\frac{k^2}{a^2}\Psi$$

Second term:

$$-\frac{\partial \Gamma^{i}_{0i}}{\partial x^{0}} = -3\frac{\partial}{\partial t}\left(\frac{1}{a}\frac{\partial a}{\partial t} + \frac{\partial \Phi}{\partial t}\right) = -3\left(\frac{\partial^2 a}{\partial t^2} - H^2 + \frac{\partial^2 \Phi}{\partial t^2}\right)$$

Third term:

$$\Gamma^{i}_{\beta i}\Gamma^{\beta}_{00} = 3H\frac{\partial \Psi}{\partial t} \quad \text{(to first order)}$$

Fourth term:

$$-\Gamma^{i}_{\beta 0}\Gamma^{\beta}_{0i} \underset{\uparrow \text{ 1st order}}{=} -\Gamma^{i}_{j0}\Gamma^{j}_{0i} = -3\left(H + \frac{\partial \Phi}{\partial t}\right)^2 = -3\left(H^2 + 2H\frac{\partial \Phi}{\partial t}\right)$$

Putting the four terms together, we obtain

$$R_{00} = -3\frac{\partial^2 a}{\partial t^2} - \frac{k^2}{a^2}\Psi - 3\frac{\partial^2 \Phi}{\partial t^2} + 3H\left(\frac{\partial \Psi}{\partial t} - 2\frac{\partial \Phi}{\partial t}\right)$$

Of course, the zeroth order term agrees with what we have calculated in the first chapter.

Now for the case when both indices are spatial, i.e.

$$R_{ij} = \frac{\partial \Gamma^\alpha_{ij}}{\partial x^\alpha} - \frac{\partial \Gamma^\alpha{}_{i\alpha}}{\partial x^j} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{ij} - \Gamma^\alpha_{\beta j}\Gamma^\beta{}_{i\alpha}$$

Again, we go term by term and distinguish between α temporal or spatial.

First term, $\alpha=0$:

$$\frac{\partial \Gamma^0_{ij}}{\partial x^0} = \delta_{ij}\frac{\partial}{\partial t}\, a^2\left[\frac{1}{a}\frac{\partial a}{\partial t} + 2\frac{1}{a}\frac{\partial a}{\partial t}(\Phi-\Psi) + \frac{\partial \Phi}{\partial t}\right]$$

$$= \delta_{ij}\left(a^2H^2 + a\frac{\partial^2 a}{\partial t^2}\right)\left[1 + 2(\Phi-\Psi)\right]$$

$$+\,\delta_{ij}\, a^2 H\left(4\frac{\partial\Phi}{\partial t} - 2\frac{\partial\Psi}{\partial t}\right) + \delta_{ij}\, a^2\frac{\partial^2\Phi}{\partial t^2}$$

Second term, $\alpha=0$:

$$-\frac{\partial \Gamma^0{}_{i0}}{\partial x^j} = -\frac{\partial}{\partial x^j}\frac{\partial\Psi}{\partial x^i} \longrightarrow k_i k_j \Psi$$

Third term, $\alpha=0$:

$$\Gamma^0_{\beta 0}\Gamma^\beta_{ij} = \underbrace{\left(\frac{\partial}{\partial t}\Psi\right)\delta_{ij}\, a^2 H}_{\beta=0}$$

$\beta=k$ term always 2nd order

Fourth term, $\alpha=0$:

$$-\Gamma^\alpha_{\beta j}\Gamma^\beta{}_{i\alpha} = \underbrace{-\frac{\partial\Psi}{\partial x^j}\frac{\partial\Psi}{\partial x^i}}_{\beta=0}\ \underbrace{-\delta_{kj}\, a^2\left[H + 2H(\Phi-\Psi) + \frac{\partial\Phi}{\partial t}\right]\delta^k_i\left(H + \frac{\partial\phi}{\partial t}\right)}_{\beta=k}$$

The $\beta=k$ term gives

$$-a^2\delta_{ij}\left[H^2 + 2H^2(\phi-\Psi) + 2H\frac{\partial\Phi}{\partial t}\right]$$

First term, $\alpha=k$:

$$\frac{\partial \Gamma^k_{ij}}{\partial x^k} \longrightarrow -\Phi\left(\delta^k_j k_i + \delta^k_i k_j - \delta_{ji} k^k\right) k^k$$

$$= -2\Phi k_i k_j + \Phi \delta_{ij} k^2$$

Second term, $\alpha = k$:

$$-\frac{\partial \Gamma^k{}_{ik}}{\partial x^j} = \Phi\left(\delta^k_i k_k + \delta^k_k k_i - \delta_{ik} k^k\right) k_j = 3\Phi k_i k_j$$

Third term, $\alpha = k$: $\beta = k$ terms are 2nd order

$$\Gamma^k{}_{0k}\Gamma^0{}_{ij} = 3\left(H + \frac{\partial \Phi}{\partial t}\right)\delta_{ij}\, a^2\left[H + 2H(\Phi - \Psi) + \frac{\partial \Phi}{\partial t}\right]$$

$$= 3a^2H^2\delta_{ij} + 6a^2H\frac{\partial \Phi}{\partial t}\delta_{ij} + 6a^2H^2(\Phi - \Psi)\delta_{ij}$$

Fourth term, $\alpha = k$: again, $\beta = k$ terms are 2nd order

$$-\Gamma^k{}_{0j}\Gamma^0{}_{ik} = -\left(H + \frac{\partial \Phi}{\partial t}\right)\delta^k_j\,\delta_{ik}\, a^2\left[H + 2H(\Phi - \Psi) + \frac{\partial \Phi}{\partial t}\right]$$

$$= -a^2H^2\delta_{ij} - 2a^2H\frac{\partial \Phi}{\partial t}\delta_{ij} - 2a^2H^2(\Phi - \Psi)\delta_{ij}$$

Putting those items together, we obtain

$$R_{ij} = \delta_{ij}\left[\left(2a^2H^2 + a\frac{d^2a}{dt^2}\right)(1 + 2\Phi - 2\Psi) + a^2H\left(6\frac{\partial \Phi}{\partial t} - \frac{\partial \Psi}{\partial t}\right) + a^2\frac{\partial^2 \Phi}{\partial t^2} + k^2\Phi\right] + k_i k_j(\Phi + \Psi)$$

We can now compute the Ricci scalar as

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij}$$

$$= (1 - 2\Psi)\left[-3\frac{\partial^2 a}{\partial t^2} - \frac{k^2}{a^2}\Psi - 3\frac{\partial^2 \Phi}{\partial t^2} + 3H\left(\frac{\partial \Psi}{\partial t} - 2\frac{\partial \Phi}{\partial t}\right)\right]$$

$$- \frac{1 - 2\Phi}{a^2}\left\{3\left[\left(2a^2H^2 + a\frac{d^2a}{dt^2}\right)(1 + 2\Phi - 2\Psi) + a^2H\left(6\frac{\partial \Phi}{\partial t} - \frac{\partial \Psi}{\partial t}\right) + a^2\frac{d^2\Phi}{dt^2} + k^2\Phi\right]\right.$$

$$\left. + k^2(\Phi + \Psi)\right\}$$

$$= -6\left(H^2 + \frac{d^2a}{dt^2}\right) + \underbrace{6\Psi \frac{1}{a}\frac{d^2a}{dt^2} - \frac{k^2}{a^2}\Psi - 3\frac{\partial^2\Phi}{\partial t^2} + 3H\left(\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Phi}{dt}\right)}_{R_{00}\text{ terms}}$$

$$\underbrace{+6\Psi\left(2H^2 + \frac{1}{a}\frac{d^2a}{dt^2}\right) - 3H\left(6\frac{\partial\Phi}{\partial t} - \frac{\partial\Psi}{\partial t}\right) - 3\frac{\partial^2\Phi}{\partial t^2} - 4\frac{k^2\Phi}{a^2} - \frac{k^2\Psi}{a^2}}_{R_{ij}\text{ terms}}$$

$$= -6\left(H^2 + \frac{d^2a}{dt^2}\right) + 12\Psi\left(H^2 + \frac{1}{a}\frac{d^2a}{dt^2}\right) - 2\frac{k^2}{a^2}\Psi - 6\frac{\partial^2\Phi}{\partial t^2} + 6H\left(\frac{\partial\Psi}{\partial t} - 4\frac{\partial\Phi}{\partial t}\right)$$
$$+ 4k^2\frac{\Phi}{a^2}$$

$$=: -6\left(H^2 + \frac{d^2a}{dt^2}\right) + \delta R$$

3.2 Evolution Equations for Φ and Ψ

We now derive two equations for Φ and Ψ from the ten $G^\mu{}_\nu = 8\pi G T^\mu{}_\nu$, i.e. we need to pick two combinations. It turns out that indeed, to linear order in Φ and Ψ the remaining equations are either zero or redundant.

We first pick the 00-component, such that we need

$$G^0{}_0 = g^{00}\left[R_{00} - \frac{1}{2}g_{00}R\right] = (1-2\Psi)R_{00} - \frac{1}{2}R$$

With our above results, we obtain for the first-order term

$$\delta G^0{}_0 = 6\Psi\frac{d^2a}{dt^2} - \frac{k^2}{a^2}\Psi - 3\frac{\partial^2\Phi}{\partial t^2} + 3H\left(\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\right)$$
$$-6\Psi\left(H^2 + \frac{1}{a}\frac{d^2a}{dt^2}\right) + \frac{k^2}{a^2}\Psi + 3\frac{\partial^2\Phi}{\partial t^2} - 3H\left(\frac{\partial\Psi}{\partial t} - 4\frac{\partial\Phi}{\partial t}\right) - 2k^2\frac{\Phi}{a^2}$$

$$= 6H\frac{\partial\Phi}{\partial t} - 6H^2\Psi + 2k^2\frac{\Phi}{a^2}$$

$G^0{}_0$ is to be equated with $8\pi G T^0{}_0$. Now, recall that $T^0{}_0$ is the energy density of all particles in the Universe.

The contribution for each species is an integral over the distribution function, such that

$$T^0{}_0(\vec{x},t) = \sum_{\text{Species } i} g_i \int \frac{d^3p}{(2\pi)^3} E_i(p) f_i(\vec{p},\vec{x},t)$$

where g_i counts the number of spin states and $E_i(p) = \sqrt{p^2 + m_i^2}$.

For CDM and baryons, we have defined the energy density as $\rho_i(1+\delta_i)$. In order to obtain T^0_0 for photons, we make use of the defining equation for Θ and write

$$T^0_0 = 2\int \frac{d^3p}{(2\pi)^3} p \left[f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Theta \right] \quad \text{(photons)}$$

The first term just the zeroth order photon density ρ_γ.

The second term yields (recall $\Theta_0(\vec{x},t) = \frac{1}{4\pi}\int d\Omega' \, \Theta(\hat{p}', \vec{x}, t)$):

$$-2\int \frac{d^3p}{(2\pi)^3} p^2 \frac{\partial f^{(0)}}{\partial p} \Theta = -\frac{2}{(2\pi)^3} \int d\Omega \int_0^\infty p^4 dp \, \frac{\partial f^{(0)}}{\partial p} \Theta$$

$$= \frac{4}{2\pi^2} \Theta_0 \; 2 \int_0^\infty p^3 dp \, f^{(0)} = 4\Theta_0 \rho_\gamma$$

such that

$$T^0{}_0 = \rho_\gamma [1 + 4\Theta_0]$$

Similarly, we can couple the neutrino perturbation to the Einstein equations via

$$T^0{}_0 = \rho_\nu [1 + 4\mathcal{N}_0]$$

Putting these things together, we obtain the first order contribution to the 00-component of the Einstein equations:

$$3H\frac{\partial \Phi}{\partial t} - 3H^2\Psi + k^2 \frac{\Phi}{a^2} = 4\pi G \left[\rho_{DM}\delta + \rho_b \delta_b + 4\rho_\gamma \Theta + 4\rho_\nu \mathcal{N}_0 \right]$$

When using conformal time η, this equation reads

$$3\frac{a'}{a}\left(\Phi' - \frac{a'}{a}\Psi\right) + k^2\Phi = 4\pi a^2 G\left[\rho_{DM}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta + 4\rho_\nu N_0\right]$$

Note that for $a = \text{const.}$, this equation reduces to the Poisson equation for gravity, $-\Delta\Phi = 4\pi G\,\delta\rho$.

We need a second evolution equation for Φ and Ψ and for this purpose, we consider the ij component of the Einstein tensor

$$G^i{}_j = g^{ik}\left[R_{kj} - \frac{1}{2}g_{kj}R\right] = -\frac{\delta^{ik}(1-2\Phi)}{a^2}R_{kj} - \frac{1}{2}\delta^i_j R$$

Using the result for R_{ij}, we see that

$$G^i{}_j = A\,\delta^i_j - \frac{k^i k_j(\Phi+\Psi)}{a^2}$$ where A is some lengthy scalar expression.

We get need to pick i and j or a particular combination of the spatial Einstein equations. Life gets particularly simple when contracting $G^i{}_j$ with the projection operator $\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j_i$, what annihilates all terms proportional to δ^i_j. This leaves us with

$$\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j_i\right)G^i{}_j = -\frac{2}{3}\frac{k^2}{a^2}(\Phi+\Psi)$$

The same contraction must be applied to the components of the energy-momentum tensor, that can be expressed in terms of the distribution functions as

$$T^{\mu\nu} = \sum_a \int \frac{d^3p}{(2\pi)^3}\, g_a \frac{P^\mu P^\nu}{E_a} f_a(\vec{p})$$

such that

$$T^i{}_j = \sum_a g_a \int \frac{d^3p}{(2\pi)^3}\frac{P^i g_{jk} P^k}{E_a} f_a(\vec{p})$$

Note:

$$\Sigma^i{}_j = \sum_a g_a \int \frac{d^3p}{(2\pi)^3}\frac{1}{E_a} f_a(\vec{p}) * \left(P^i P_j - \frac{1}{3}p^2\delta^i{}_j\right)$$

is called the anisotropic stress

We also recall the definition from Chapter 2 $p^2 := -g_{ij} P^i P^j$ and obtain

$$(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta^j_i)\, T^i{}_j = -\sum_a g_a \int \frac{d^3p}{(2\pi)^3} \frac{p^2\mu^2 - \frac{1}{3}p^2}{E_a} f_a(\vec{p})$$

We recognize here the second Legendre polynomial $P_2(\mu) = \frac{3\mu^2-1}{2}$, what implies that we pick out the quadrupoles that can be non-zero only for photons and for neutrinos.

The integral for the photons gives us

$$-2\int_0^\infty \frac{p^2 dp}{(2\pi)^2}\, p \int_{-1}^{1} d\mu\, \frac{2}{3} P_2(\mu) f = \frac{2}{3\pi^2}\int_0^\infty dp\, p^4 \frac{\partial f^{(0)}}{\partial p} \int_{-1}^{1} \frac{d\mu}{2} P_2(\mu)\Theta(\mu)$$

$$f = f^{(0)} - p\frac{\partial}{\partial p} f^{(0)}\Theta$$

$$= -\frac{2}{3\pi^2}\Theta_2 \int_0^\infty p^4\, dp\, \frac{\partial f^{(0)}}{\partial p}$$

$$\Theta_\ell = \frac{1}{(-i)^\ell}\int_{-1}^{1} \frac{d\mu}{2} P_\ell(\mu)\Theta(\mu)$$

$$= \frac{8}{3\pi^2}\Theta_2 \underbrace{\int_0^\infty p^3\, dp\, f^{(0)}}_{= \frac{\pi^4 T^4}{15} = \pi^2 \rho_\gamma} = \frac{8\rho_\gamma \Theta_2}{3}$$

The neutrinos can be treated accordingly. In total, we obtain the equation

$$k^2(\Phi + \Psi) = -32\pi G a^2 [\rho_\gamma \Theta_2 + \rho_\nu \mathcal{N}_2]$$

Since Compton scattering suppresses the photon multipoles, only the neutrinos act as a source in this equation at early times. In the absence of quadrupoles Φ and Ψ are equal and opposite.

3.3 Tensor Perturbations

$\Phi(\vec{x},t)$ and $\Psi(\vec{x},t)$ remain unchanged under spatial coordinate transformations, i.e. they are scalars. As we have seen, they couple therefore directly to the density perturbations. However, there may as well be tensor perturbations in the metric. These would cause a so-called B-mode polarisation signal in the CMB. The detection of B-modes was recently announced by the BICEP2 collaboration and presently causes a lot of excitement.

The metric of tensor perturbations in the x-y plane can be parametrized choosing $g_{00}=1$, $g_{0i}=0$ and

$$g_{ij} = -a^2 \begin{pmatrix} 1+h_+ & h_\times & 0 \\ h_\times & 1-h_+ & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The functions h_+ and h_- are the two independent components of a 3-dimensional rank-2 tensor that is divergenceless and traceless. When we call this tensor

$$\mathcal{H}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow g_{ij} = -a^2(\delta_{ij} + \mathcal{H}_{ij})$$

this means $k^i \mathcal{H}_{ij} = k^j \mathcal{H}_{ij} = 0$, what implies that above choice corresponds to setting $\vec{k} = \hat{z}$.

In order to derive evolution equations for the tensor modes, we apply the same straightforward approach as for scalars, i.e. we first derive the Christoffel symbols. We once again recall their form

$$\Gamma^\kappa_{\mu\nu} = \frac{g^{\kappa\alpha}}{2}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\alpha} \right)$$

The only non-zero terms now come from derivatives of g_{ij}, such that we immediately see that $\Gamma^0_{00} = \Gamma^0_{i0} = \Gamma^i_{00} = 0$.

The symbol with two lower spatial indices is

$$\Gamma^0_{ij} = -\frac{g^{00}}{2}\frac{\partial g_{ij}}{\partial x^0} = -\frac{1}{2}\frac{\partial g_{ij}}{\partial t} = -\frac{1}{2}\left(2H g_{ij} - a^2 \frac{\partial \mathcal{H}_{ij}}{\partial t}\right)$$

$$= -H g_{ij} + a^2 \frac{\partial \mathcal{H}_{ij}}{\partial t}$$

Furthermore, we find

$$\Gamma^i_{0j} = \frac{g^{ik}}{2}\frac{\partial g_{jk}}{\partial t} = \frac{g^{ik}}{2}\left(2Hg_{jk} - a^2 \frac{\partial \mathcal{H}_{jk}}{\partial t}\right) = H\delta^i_j + \frac{1}{2}\frac{\partial \mathcal{H}_{ij}}{\partial t}$$

and

$$\Gamma^i_{jk} = \frac{\overset{\nearrow \frac{1}{a^2}}{g^{il}}}{2}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

$$= -\frac{\delta^{il}}{2a^2}(-a^2)\left(ik_j\mathcal{H}_{kl} + ik_k\mathcal{H}_{jl} - k_l\mathcal{H}_{jk}\right)$$

$$= \frac{i}{2}\left(k_k\mathcal{H}_{ij} + k_j\mathcal{H}_{ik} - k_i\mathcal{H}_{jk}\right)$$

Now, the time-time component of the Ricci tensor is

$$R_{00} = \frac{\partial \overset{\nearrow 0}{\Gamma^\alpha_{00}}}{\partial x^\alpha} - \frac{\partial \Gamma^\alpha_{0\alpha}}{\partial x^0} + \Gamma^\alpha_{\beta\alpha}\overset{\nearrow 0}{\Gamma^\beta_{00}} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} = -\frac{\partial \Gamma^i_{0i}}{\partial x^0} - \Gamma^i_{j0}\Gamma^j_{0i}$$

$\sum_i \mathcal{H}_{ii} = 0$

$$= -3\frac{\partial H}{\partial t} - \left(H\delta_{ij} + \frac{1}{2}\frac{\partial \mathcal{H}_{ij}}{\partial t}\right)\left(H\delta_{ij} + \frac{1}{2}\frac{\partial \mathcal{H}_{ij}}{\partial t}\right)$$

$$= -3\frac{\partial H}{\partial t} - 3H^2 = -3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} - 3\frac{\dot{a}^2}{a^2} = -3\frac{1}{a}\frac{d^2 a}{dt^2}$$

Now, this confirms our earlier result, but moreover, we observe that the first-order tensor contributions to R_{00} cancel. It will turn out that the same is true for R, but not for R_{ij}. This tells us, that density perturbations, that form the right-hand side of the Einstein equations induce scalar metric perturbations Φ and Ψ but no tensors $\mathcal{H}$. Tensor perturbations therefore decouple, which is an aspect of the so-called decomposition theorem that we discuss below.

But now for the spatial components, that are given by

$$R_{ij} = \frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}_{i\alpha}}{\partial x^{j}} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{ij} - \Gamma^{\alpha}_{\beta j}\Gamma^{\beta}_{i\alpha}$$

The first two terms together yield

$$\Gamma^{k}_{ij} = \frac{i}{2}\left(k_j \mathcal{H}_{ki} + k_i \mathcal{H}_{kj} - k_k \mathcal{H}_{ij}\right)$$

$$\Gamma^{k}_{ik} = \frac{i}{2}\Big(\cancel{k_k \mathcal{H}_{ki}} + \underbrace{k_i \mathcal{H}_{kk}}_{=0} - \cancel{k_k \mathcal{H}_{ik}}\Big)$$

$$\frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}_{i\alpha}}{\partial x^{j}} = \frac{\partial \Gamma^{0}_{ij}}{\partial t} + \frac{\partial \Gamma^{k}_{ij}}{\partial x^{k}} - \frac{\partial \Gamma^{k}_{ik}}{\partial x^{j}}$$

$$= -\frac{1}{2}\frac{\partial^2 g_{ij}}{\partial t^2} - \frac{1}{2}\Big(\underbrace{k_j k_k \mathcal{H}_{ki} + k_i k_j \mathcal{H}_{kj}}_{=0\ (\text{divergenceless})} - k^2 \mathcal{H}_{ij}\Big)$$

$$= -\frac{1}{2}\frac{\partial^2 g_{ij}}{\partial t^2} + \frac{k^2}{2}\mathcal{H}_{ij}$$

For the third term, we find

$$\Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{ij} = \Gamma^{k}_{0k}\Gamma^{0}_{ij} + \underbrace{\Gamma^{k}_{lk}\Gamma^{l}_{ij}}_{\text{2nd order}} = -\frac{3}{2}H\frac{\partial g_{ij}}{\partial t}$$

and for the fourth

$$-\Gamma^{\alpha}_{\beta j}\Gamma^{\beta}_{i\alpha} = -\Gamma^{k}_{0j}\Gamma^{0}_{ik} - \Gamma^{0}_{kj}\Gamma^{k}_{i0}$$

$$= -a^2\left(H\delta_{kj} + \frac{1}{2}\frac{\partial \mathcal{H}_{kj}}{\partial t}\right)\left(H\delta_{ik} + H\mathcal{H}_{ik} + \frac{1}{2}\frac{\partial \mathcal{H}_{ik}}{\partial t}\right)$$

$$-a^2\left(H\delta_{ik} + \frac{1}{2}\frac{\partial \mathcal{H}_{ik}}{\partial t}\right)\left(H\delta_{kj} + H\mathcal{H}_{kj} + \frac{1}{2}\frac{\partial \mathcal{H}_{kj}}{\partial t}\right)$$

$$= 2H^2 g_{ij} - 2a^2 H\frac{\partial \mathcal{H}_{ij}}{\partial t}$$

In total, we obtain

$$R_{ij} = -\frac{1}{2}\frac{\partial^2 g_{ij}}{\partial t^2} + \frac{k^2}{2}\mathcal{H}_{ij} - \frac{3}{2}H\frac{\partial g_{ij}}{\partial t} + 2H^2 g_{ij} - 2a^2 H\frac{\partial \mathcal{H}_{ij}}{\partial t}$$

The time derivatives are given by

$$\frac{\partial g_{ij}}{\partial t} = \left(2Hg_{ij} - a^2 \frac{\partial \mathcal{H}_{ij}}{\partial t}\right)$$

$$\frac{\partial^2 g_{ij}}{\partial t^2} = 2g_{ij}\left(\frac{1}{a}\frac{d^2a}{dt^2} - H^2\right) + 2H\left(2Hg_{ij} - a^2\frac{\partial \mathcal{H}_{ij}}{\partial t}\right) - 2a^2H\frac{\partial \mathcal{H}_{ij}}{\partial t} - a^2\frac{\partial^2 \mathcal{H}_{ij}}{\partial t^2}$$

$$= 2g_{ij}\left(\frac{1}{a}\frac{d^2a}{dt^2} + H^2\right) - 4a^2H\frac{\partial \mathcal{H}_{ij}}{\partial t} - a^2\frac{\partial^2 \mathcal{H}_{ij}}{\partial t^2}$$

such that

$$R_{ij} = -g_{ij}\left(\frac{1}{a}\frac{d^2a}{dt^2} + 2H^2\right) + \frac{3}{2}a^2H\frac{\partial \mathcal{H}_{ij}}{\partial t} + \frac{1}{2}a^2\frac{\partial^2 \mathcal{H}_{ij}}{\partial t^2} + \frac{k^2}{2}\mathcal{H}_{ij}$$

We can now obtain the Ricci scalar as

$$R = g^{00}R_{00} + g^{ij}R_{ij} = -3\frac{1}{a}\frac{d^2a}{dt^2} - 3\frac{1}{a}\frac{d^2a}{dt^2} - 6H^2 = -6\frac{\ddot{a}}{a} - 6\frac{\dot{a}^2}{a^2}$$

Which agrees with what we have stated above (independence of tensor perturbations at first order) and with what we have derived earlier.

The first-order Einstein tensor is $\delta G^i{}_j = \delta R^i{}_j$ ($\delta G^0{}_0 = 0$), and explicitly

$$\delta G^i{}_j = \frac{3}{2}H\frac{\partial \mathcal{H}_{ij}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{ij}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{ij}$$

From the Einstein equations, we want to derive equations for the independent components h_+ and $h_\times$. An equation for h_+ can be obtained from

$$\delta G^1{}_1 - \delta G^2{}_2 = 3H\frac{\partial h_+}{\partial t} + \frac{\partial^2 h_+}{\partial t^2} + \frac{k^2}{a^2}h_+$$

$$\Longrightarrow \quad dt = a\,d\eta \qquad \frac{\partial}{\partial t} = \frac{1}{a}\frac{\partial}{\partial \eta} \qquad \frac{\partial^2}{\partial t^2} = -\frac{a'}{a^3}\frac{\partial}{\partial \eta} + \frac{1}{a^2}\frac{\partial^2}{\partial \eta^2}$$

$$a^2(\delta G^1{}_1 - \delta G^2{}_2) = h_+'' - 2\frac{a'}{a}h_+' + k^2 h_+$$

Now, when $\Theta = \Theta(p,\mu)$, then $T^1{}_1 = T^2{}_2$. If Θ would also depend on the polar angle φ, this would not be true. In turn, anisotropies induced by tensor perturbations take the form $\Theta(p, \mu, \varphi)$. It turns out however that these have a negligible impact on the evolution of $h_{+,\times}$, such that we can set the right-hand side of their evolution equation to zero.

From $\delta G^1{}_2$, we see that $h_\times$ observes the same equation as h_+. Note that $k_1 = k_2 = 0$, such that the scalar term $-\frac{1}{a^2} k_i k_j (\Phi + \Psi)$ does not contribute. To summarize, we find $h_\alpha'' - 2\frac{a'}{a} h_\alpha' + k^2 h_\alpha = 0$ where $\alpha = +, \times$.

Solutions to this equation are called gravity waves. In a matter or radiation dominated Universe, they decay when their wave-length becomes smaller than the horizon. For the CMB, this implies that we can observe only the imprint of gravity waves that were of horizon size or larger at the time of decoupling (of electrons and photons).

Now let us briefly describe the decomposition theorem, which states that any perturbation of scalar, vector or tensor type does not induce perturbations of either of the two other categories.

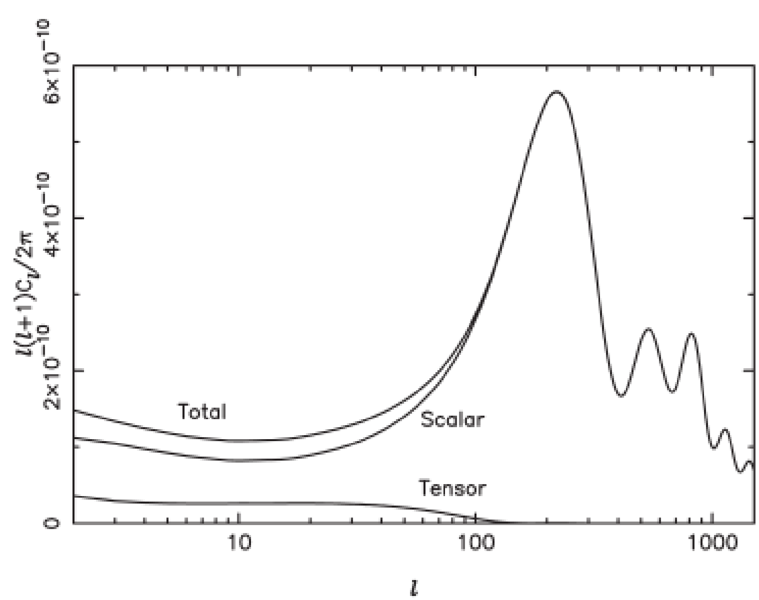

Fig. 12.3. The scalar and tensor temperature anisotropies for chaotic inflation with a quadratic potential, corresponding to tensor fraction $r = 0.16$.

from Liddle & Lyth

We have already observed that the scalar perturbations do not feed into the evolution equations for gravity waves. Now for the scalar perturbations, we have derived evolution equations from the Einstein tensor components $G^0{}_0$ and $(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})G^i{}_j$. Clearly, the tensors do not feed into $G^0{}_0$, because we have seen that $R^0{}_0$ and R do not depend on the tensor perturbations. Now for the remaining equation, we note that

$$(\hat{k}_i\hat{k}_j - \tfrac{1}{3}\delta_{ij})\delta G^i{}_j$$

$$=(\delta_{i3}\delta_{j3} - \tfrac{1}{3}\delta_{ij})\left(\frac{3}{2}H\frac{\partial \mathcal{H}_{ij}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{ij}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{ij}\right)$$

what vanishes due to the form of $\mathcal{H}_{ij}$.

3.4 Gauge Transformations

The way in which we have described the scalar metric perturbations so far is called the conformal Newtonian gauge. Now, in General Relativity, we can perform coordinate transformations and end up that way in different gauges. There can be technical advantages to this, for example, in calculations on inflation a spatially flat slicing with g_{ij} unperturbed is often used. Boltzmann codes on the CMB are typically numerically best behaved in the so-called synchronous gauge. It is hence useful to be able to move back and forth between the different gauges.

The most general scalar perturbations to the metric can be expressed in terms of four scalar functions

A, B, ψ and E as

$$g_{00} = 1 + 2A$$
$$g_{0i} = -a \frac{\partial B}{\partial x^i}$$
$$g_{ij} = -a^2 \left(\delta_{ij} [1 + 2\psi] - 2 \frac{\partial^2 E}{\partial x^i \partial x^j} \right)$$

Conformal Newtonian gauge corresponds to $A = \Psi$, $\psi = \Phi$ and $B = E = 0$.

A coordinate transformation requires a change of the metric such that the invariant distance is preserved:

$$\tilde{g}_{\alpha\beta}(\tilde{x}) d\tilde{x}^\alpha d\tilde{x}^\beta = g_{\mu\nu}(x) dx^\mu dx^\nu, \text{ where } d\tilde{x}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} dx^\mu$$

$$\Longrightarrow \tilde{g}_{\alpha\beta}(\tilde{x}) \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} = g_{\mu\nu}(x)$$

A general coordinate transformation is generated by

$$t \longmapsto \tilde{t} = t + \xi^0(t, \vec{x}) \quad \text{and} \quad x^i \longmapsto \tilde{x}^i = x^i + \delta^{ij} \frac{\partial \xi(t, \vec{x})}{\partial x^j}$$

We can stick with this linearised form by taking ξ^0 and ξ to be of the same order as the perturbation variables.

Now, for example, the 00 component of the transformation formula is given by

$$\tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^\alpha}{\partial t} \frac{\partial \tilde{x}^\beta}{\partial t} = 1 + 2A$$

Consider first $\alpha = 0$, $\beta = i$. $\tilde{g}_{0i} = -a \frac{\partial B}{\partial \tilde{x}^i}$ is a first order perturbation and also $\frac{\partial \tilde{x}^i}{\partial t} = \frac{\partial^2 \xi}{\partial x^j \partial t}$ is first order such that this contribution can be neglected. Likewise, $\alpha = i$, $\beta = 0$ and $\alpha = i$, $\beta = j$ do not contribute at first order. To first order, the left hand side of above equation therefore simply reads

$$(1+2\tilde{A})\left(\frac{\partial\tilde{t}}{\partial t}\right)^2=(1+2\tilde{A})\left(1+\frac{\partial\xi^0}{\partial t}\right)^2=1+2\tilde{A}+2\frac{\partial\xi^0}{\partial t}$$

The coordinate transformation hence leads to

$$A\longmapsto\tilde{A}=A-\frac{\partial\xi^0}{\partial t}$$

Using similar arguments, we can derive the remaining transformations:

0i component:

$$\tilde{g}_{\alpha\beta}\frac{\partial\tilde{x}^\alpha}{\partial t}\frac{\partial\tilde{x}^\beta}{\partial x^i}=-a\frac{\partial B}{\partial x^i}=-a\frac{\partial\tilde{B}}{\partial x^i}+\frac{\partial\tilde{x}^0}{\partial x^i}-a^2\frac{\partial\tilde{x}^i}{\partial t}=-a\frac{\partial\tilde{B}}{\partial x^i}+\frac{\partial\xi^0}{\partial x^i}-a^2\frac{\partial^2\xi}{\partial x^i\partial t}$$

$$\Longrightarrow B\longmapsto\tilde{B}=B+\frac{\xi^0}{a}-a\frac{\partial\xi}{\partial t}$$

$i\neq j$ component:

$$\tilde{g}_{\alpha\beta}\frac{\partial\tilde{x}^\alpha}{\partial x^i}\frac{\partial\tilde{x}^\beta}{\partial x^j}=2a^2\frac{\partial^2E}{\partial x^i\partial x^j}=\underbrace{2a^2\frac{\partial^2\tilde{E}}{\partial x^i\partial x^j}}_{\alpha=i\;\;\beta=j}-\underbrace{2a^2\frac{\partial^2\xi}{\partial x^i\partial x^j}}_{\substack{\alpha=\beta=i\\ \alpha=\beta=j}}$$

$$\Longrightarrow E\longmapsto\tilde{E}=E+\xi$$

ii component:

$$\tilde{g}_{\alpha\beta}\frac{\partial\tilde{x}^\alpha}{\partial x^i}\frac{\partial\tilde{x}^\beta}{\partial x^i}=-a^2(t)\left([1+2\psi]-2\frac{\partial^2E}{\partial x^i\partial x^i}\right)$$

$$=\tilde{g}_{ii}\frac{\partial\tilde{x}^i}{\partial x^i}\frac{\partial\tilde{x}^i}{\partial x^i}=-a^2(\tilde{t})\left([1+2\tilde{\psi}]-2\frac{\partial^2\tilde{E}}{\partial x^{i2}}\right)\left(1+\frac{\partial^2\xi}{\partial x^{i2}}\right)^2$$

$$=-a^2(\tilde{t})(1+2\tilde{\psi})+2a^2\frac{\partial^2E}{\partial x^{i2}}+2a^2\frac{\partial^2\xi}{\partial x^{i2}}-2a^2\frac{\partial^2\xi}{\partial x^{i2}}$$

$$a(\tilde{t})=a(t+\xi^0)=a+\dot{a}\xi_0=a(1+H\xi_0)$$

$$\Longrightarrow \psi\longmapsto\tilde{\psi}-H\xi_0$$

Now there should be two gauge invariant variables, and indeed, these are given by the Bardeen potentials

$$\Phi_A = A + \frac{1}{a}\frac{\partial}{\partial \eta}\left[a(\dot{E} - B)\right]$$

$$\Phi_H = -\psi + aH(B - \dot{E})$$

In conformal Newtonian gauge, $\Phi_A = \Psi$ and $\Phi_H = -\Phi$.
If we find equations in one gauge, we can then easily transform these to a different gauge via these gauge invariant variables, without working out the tedious coordinate transformations directly.
Similarly, there are certain gauge invariant combinations of $T_{\mu\nu}$:

$$v = ikB - \frac{\hat{k}^i T^0{}_i}{a(\rho + P)}$$

For pressureless matter, this agrees with the v defined earlier in conformal Newtonian gauge. For radiation, $v = -3i\Theta_1$.
Another invariant is the generalised perturbation in energy density

$$\varepsilon_m = -1 + \frac{T^0{}_0}{\rho} - \frac{3H}{k^2 \rho} k^i T^0{}_i .$$

For pressureless matter in conformal Newtonian gauge,

$$\varepsilon_m = \delta + \frac{3aHv}{k}$$ and for radiation $$\varepsilon_m = 4\Theta_0 - 12i\Theta_1 \frac{aH}{k}.$$

These expressions reduce to the standard overdensities on subhorizon scales $k \gg aH$.

3.5 Summary

The evolution equations for the scalar metric perturbations are given by

$$3\frac{a'}{a}\left(\Phi' - \frac{a'}{a}\Psi\right) + k^2\Phi = 4\pi a^2 G\left[\rho_m\delta_m + \rho_b\delta_b + 4\rho_r\Theta_r\right]$$

$$k^2(\Phi + \Psi) = -32\pi G a^2 \rho_r \Theta_{r,2}$$

The subscript m includes the matter components (i.e. baryons and CDM) and r the radiation components (i.e. photons and neutrinos):

$$\rho_m\delta_m = \rho_{DM}\delta + \rho_b\delta_b \,, \qquad \rho_r\Theta_{r,0} = \rho_\gamma\Theta_0 + \rho_\nu\mathcal{N}_0$$

$$\rho_m v_m = \rho_{DM} v + \rho_b v_b \,, \qquad \rho_r\Theta_{r,1} = \rho_\gamma\Theta_1 + \rho_\nu\mathcal{N}_1$$

Tensor perturbations and scalar perturbations decouple, and the former evolve corresponding to

$$h''_\alpha + 2\frac{a'}{a}h'_\alpha + k^2 h_\alpha = 0 \quad \text{where } \alpha = +, \times$$

The amplitude of a gravity wave decays upon horizon entry.