2. Boltzmann Equations

One of the key differences between Baryons and Cold Dark Matter is that the former undergo frequent interactions before hydrogen recombines in the Early Universe. This leaves a distinct imprint in the CMB and the LSS, what can be described theoretically with Boltzmann equations. Yet, Dark Matter influences baryons through gravitational interactions, that leave an imprint in the metric.

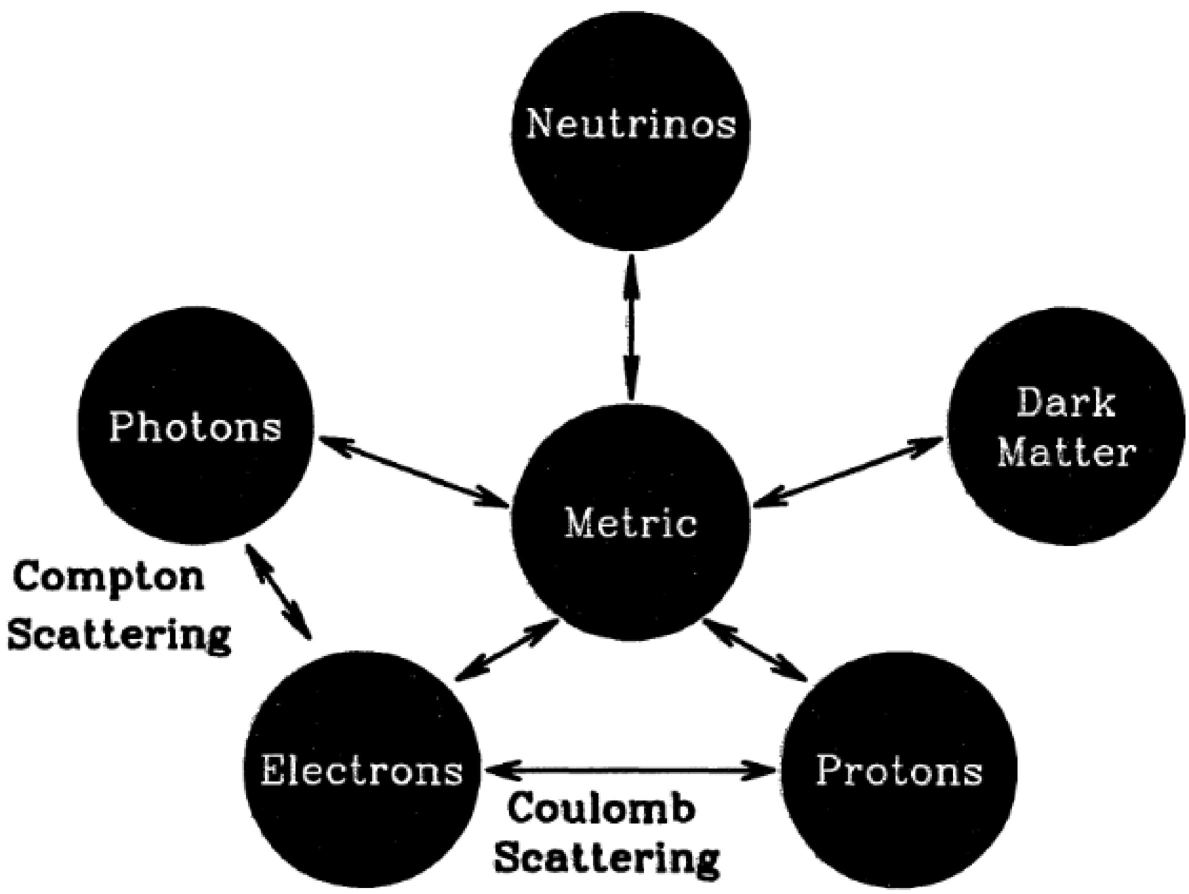


Figure 4.1. The ways in which the different components of the universe interact with each other. These connections are encoded in the coupled Boltzmann–Einstein equations.

Above Figure illustrates which effects the Boltzmann equations that we derive here are supposed to describe.

A Boltzmann equation takes the general form

$$\frac{d f(x,p)}{dt} = \mathcal{C}[f](x,p)$$

The distribution functions of the fluid components are denoted by f, and $\mathcal{C}$ is called the collision term.

2.1 The Collisionless Boltzmann Equation for Photons

We first need to specify the meaning of the left-hand side of the Boltzmann equation within the expanding background, also accounting for metric perturbations. For this purpose, we introduce the Newtonian potential $\Psi(\vec{x},t)$ and the perturbation of the spatial curvature $\Phi(\vec{x},t)$. In terms of these, we express the metric as

$$g_{00}(\vec{x},t) = 1 + 2\Psi(\vec{x},t)$$
$$g_{0i}(\vec{x},t) = 0$$
$$g_{ij}(\vec{x},t) = -a^2 \delta_{ij}\left(1+2\Phi(\vec{x},t)\right)$$

This parametrisation is called the conformal Newtonian gauge, as in General Relativity, coordinate transformation correspond to gauge transformations. Moreover, we account here only for scalar perturbations, whereas in general, one may also expect vector and tensor perturbations. The latter have however not been detected until present. We will return to these matters later, if time permits, and proceed with conformal Newtonian gauge for now. Next, we need to express the total time derivative on the left hand side of the Boltzmann equation in terms of partial derivatives with respect to the coordinates x^μ and the four-momentum

$$P^\mu = \frac{dx^\mu}{d\lambda},$$

where λ is a parameter for the path, and for the massless photon, we require that

$$P^2 = g_{\mu\nu} P^\mu P^\nu = 0$$

With the explicit form of the metric above, we obtain

$$P^2 = (1+2\Psi) P^{0^2} - p^2 = (1+2\Psi) P^{0^2} - a^2 P^i P^i (1+2\Phi) = 0$$

where $p^2 := -g_{ij} P^i P^j$

$$\Rightarrow \quad P^0 = \frac{p}{\sqrt{1+2\Psi}} = p(1-\Psi) + \mathcal{O}(\Psi^2)$$

In the following, we aim for first order accuracy and tacitly drop $\mathcal{O}(\Psi^2, \Phi^2)$ terms (even when using $=$ instead of $\simeq$).

Note that an overdense (underdense) region has $\Psi < 0$ ($\Psi > 0$). Therefore, a photon in an overdense region has $P^0 > p$ and it will lose energy, i.e. they experience redshift, when moving out of that region.

In order to express the total time derivative, we use above relation to eliminate P^0 in favour of p. Besides the magnitude p, there is also a dependence on the direction of the momentum $\hat{p}^i$, that we normalise as $\delta_{ij} \hat{p}^i \hat{p}^j = 1$. We can then express

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial t} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial f}{\partial \hat{p}^i} \frac{\partial \hat{p}^i}{\partial t}$$

Now, we aim to derive equations that are accurate to first order in perturbation theory. At zeroth order, f is given by the Bose Einstein distribution, that is direction independent. Hence, $\frac{\partial f}{\partial \hat{p}^i}$ is first order and so is $\frac{\partial \hat{p}^i}{\partial t}$, as a photon only changes its direction in the presence of a perturbation potential. Hence, the last term is of second order in the perturbations or smaller, and we can neglect it.

Now for the term $\frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial t}$. Recall that $P^i = \frac{dx^i}{d\lambda}$ and $P^0 = \frac{dt}{d\lambda}$. It follows that

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{P^i}{P^0}$$

Now, we want to express P^i through p and $\hat{p}^i$. Since the perturbations are scalar, we should have that $P^i = C\hat{p}^i$.

$$\Rightarrow p^2 = -g_{ij}\hat{p}^i\hat{p}^j C^2 = a^2(1+2\Phi)\delta_{ij}\hat{p}^i\hat{p}^j C^2 = a^2(1+2\Phi)C^2$$

$$\Rightarrow C = \frac{1}{a}(1-\Phi)p \quad \text{(to first order)}$$

$$\Rightarrow P^i = p\hat{p}^i\frac{1-\Phi}{a}$$

$$\Rightarrow \frac{dx^i}{dt} = \frac{p\hat{p}^i(1-\Phi)}{ap(1-\Psi)} \approx \frac{\hat{p}^i}{a}(1+\Psi-\Phi)$$

Now, since $\frac{\partial f}{\partial x^i}$ is already of first order, we can neglect the potential terms and find up to this point

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial t}$$

Calculating $\frac{\partial p}{\partial t}$ causes more work, as we need to make use of the geodesic equation. We start with recalling that

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} \Longrightarrow \frac{dP^0}{d\lambda} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta$$

$$\frac{d}{d\lambda} = \frac{dt}{d\lambda}\frac{d}{dt} = P^0\frac{d}{dt} \Longrightarrow \frac{d}{dt}[p(1-\Psi)] = -\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+\Psi)$$

$$\Longrightarrow \frac{dp}{dt}(1-\Psi) = p\frac{d\Psi}{dt} - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+\Psi)$$

$$\overset{*(1+\Psi)}{\Longrightarrow} \frac{dp}{dt} = p\left(\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right) - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi)$$

Now for the last term:

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{g^{0\nu}}{2}\left[2\frac{\partial g_{\nu\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right]\frac{P^\alpha P^\beta}{p}$$

Using the perturbed metric in conformal Newtonian gauge, we see that $\nu=0$ and

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{1-2\Psi}{2}\left[4\frac{\partial\Psi}{\partial x^\beta}\delta^0_\alpha - \frac{\partial g_{\alpha\beta}}{\partial t}\right]\frac{P^\alpha P^\beta}{p}$$

We further evaluate

$$-\frac{\partial g_{\alpha\beta}}{\partial t}\frac{P^\alpha P^\beta}{p} = -\frac{\partial g_{00}}{\partial t}\frac{P^0 P^0}{p} - \frac{\partial g_{ij}}{\partial t}\frac{P^i P^j}{p}$$

$$= -\frac{2\partial\Psi}{\partial t}p + a^2\delta_{ij}\left[2\frac{\partial\Phi}{\partial t} + 2H(1+2\Phi)\right]\frac{P^i P^j}{p}$$

$$= -2\frac{\partial\Psi}{\partial t}p + \left[2\frac{\partial\Phi}{\partial t} + 2H(1+2\Phi)\right]p(1-2\Phi)$$

$$\delta_{ij}P^iP^j = \frac{1}{a^2}p^2(1-2\Phi)$$

$$= -2p\left[\frac{\partial\Psi}{\partial t} - \frac{\partial\Phi}{\partial t} - H\right]$$

such that we obtain

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{1-2\Psi}{2}\left[4\frac{\partial\Psi}{\partial x^\beta}P^\beta - 2p\frac{\partial\Psi}{\partial t} + 2p\frac{\partial\Phi}{\partial t} + 2pH\right]$$

$$= (1-2\Psi)\left[2\left(\frac{\partial\Psi}{\partial t}p + \frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a}\right) - p\left(\frac{\partial\Psi}{\partial t} - \frac{\partial\Phi}{\partial t} - H\right)\right]$$

$$= (1-2\Psi)\left[\frac{\partial\Psi}{\partial t}p + 2\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a} + p\left(\frac{\partial\Phi}{\partial t} + H\right)\right]$$

and

$$\frac{dp}{dt} = p\left(\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right) - \frac{\partial\Psi}{\partial t}p - 2\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a} - p\left(\frac{\partial\Phi}{\partial t} + H\right)$$

$\Rightarrow$

$$\frac{1}{p}\frac{dp}{dt} = -H - \frac{\partial \Phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}$$

The first term of this equation accounts for the red-shift of the photon due to Hubble expansion. Regarding the other terms, note that in an overdense region, $\Phi > 0$ and $\Psi < 0$. In a deepening gravitational well, $\left(\frac{\partial \Phi}{\partial t} > 0\right)$ a photon loses energy, likewise, when travelling outside of a well $\left(\hat{p}^i \frac{\partial \Psi}{\partial x^i} > 0\right)$.

Putting everything together, the collisionless Boltzmann equation for photons is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right]$$

These equations are suitable in order to solve for first-order perturbations in the photon fluid, that we parametrise as

$$f(\vec{x}, p, \hat{p}, t) = \frac{1}{e^{\frac{p}{T(t)[1+\Theta(\vec{x}, \hat{p}, t)]}} - 1}$$

$$= f^{(0)} - p\frac{\partial}{\partial p} f^{(0)} \Theta \quad \text{with} \quad f^{(0)} = \frac{1}{e^{\frac{p}{T}} - 1}$$

Now we plug this ansatz into the Boltzmann equation and first only keep the leading (zeroth order) terms:

$$\left.\frac{df}{dt}\right|_0 = \frac{\partial f^{(0)}}{\partial t} - Hp\frac{\partial f^{(0)}}{\partial p} = 0$$

Note that even if we were including collisions, the collision term would be zero at zeroth order, as it should vanish for equilibrium distributions.

Note that this equation implies the dependence of the photon temperature on the scale factor:

$$\frac{df^{(0)}}{dt} = \frac{\partial f^{(0)}}{\partial T}\frac{dT}{dt} = -\frac{dT}{dt}\frac{1}{T}\,p\,\frac{\partial f^{(0)}}{\partial p}$$

$$\Longrightarrow \left[-\frac{dT}{dt}\frac{p}{T} - \frac{da}{dt}\frac{p}{a}\right]\frac{\partial f^{(0)}}{\partial p} = 0$$

$$\Longrightarrow \frac{dT}{T} = -\frac{da}{a} \Longrightarrow T \propto \frac{1}{a}$$

Now for the first order terms:

$$\left.\frac{df}{dt}\right|_1 = -p\frac{\partial}{\partial t}\left[\frac{\partial f^{(0)}}{\partial p}\Theta\right] - p\frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i}\frac{\partial f^{(0)}}{\partial p} + Hp\,\Theta\frac{\partial}{\partial p}\left[p\frac{\partial f^{(0)}}{\partial p}\right]$$
$$- p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right]$$

We rewrite the first term as

$$-p\frac{\partial}{\partial t}\left[\frac{\partial f^{(0)}}{\partial p}\Theta\right] = -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial \Theta}{\partial t} - p\,\Theta\frac{dT}{dt}\frac{\partial^2 f^{(0)}}{\partial T\,\partial p}$$
$$= -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial \Theta}{\partial t} + p\,\Theta\frac{1}{T}\frac{dT}{dt}\frac{\partial}{\partial p}\left[p\frac{\partial f^{(0)}}{\partial p}\right]$$

$$\frac{\partial f^{(0)}}{\partial T} = -\frac{p}{T}\frac{\partial f^{(0)}}{\partial p}$$

Now, $\frac{\dot{T}}{T} = -\frac{\dot{a}}{a} = -H$ such that

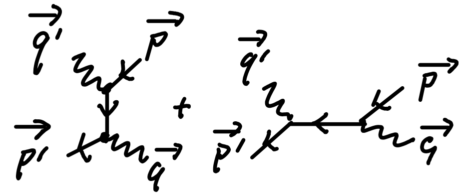
$$\left.\frac{df}{dt}\right|_1 = -p\frac{\partial f^{(0)}}{\partial p}\left[\underbrace{\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i}}_{\text{"free streaming"}} + \underbrace{\frac{d\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}}_{\text{gravitational interactions}}\right]$$

The particular terms in this equation can be interpreted as due to the "free streaming" of the over- and under-densities and as due to gravitational potentials as it is indicated.

2.2 Boltzmann Equation for Photons with Collisions

Now, we extend the Boltzmann equations to include the effect of Compton scattering,

$$e^-(\vec{q}) + \gamma(\vec{p}) \longleftrightarrow e^-(\vec{q}\,') + \gamma(\vec{p}\,')$$



The collision term for photons from Compton scattering is

$$C[f(\vec{p})] = \frac{1}{|\vec{p}|}\int \frac{d^3q}{(2\pi)^3 2E_e(\vec{q})}\int \frac{d^3p'}{(2\pi)^3 2E(\vec{p}\,')}\int \frac{d^3q'}{(2\pi)^3 2E_e(\vec{q}\,')}$$

$$* (2\pi)^4 \delta^3(\vec{p}+\vec{q}-\vec{p}\,'-\vec{q}\,')\,\delta\big(E(\vec{p})+E_e(\vec{q})-E(\vec{p}\,')-E_e(\vec{q}\,')\big)\,|\mathcal{M}|^2\,\mathcal{F}$$

where $E(\vec{p}) = |\vec{p}|$, $E_e(\vec{q}) = \sqrt{\vec{q}^{\,2}+m_e^2}$

$$\mathcal{F} = (1+f(\vec{p}))(1-f_e(\vec{q}))f(\vec{p}\,')f_e(\vec{q}\,') - f(\vec{p})f_e(\vec{q})(1+f(\vec{p}\,'))(1-f_e(\vec{q}\,'))$$

Now, during the epoch of interest, $m_e \gg T$, such that

$$E_e(\vec{q}) \approx m_e + \frac{\vec{q}^{\,2}}{2m_e}$$

and we can neglect all Pauli-Blocking terms:

$$\mathcal{F} \approx (1+f(\vec{p}))f(\vec{p}\,')f_e(\vec{q}\,') - f(\vec{p})f_e(\vec{q})(1+f(\vec{p}\,'))$$

The terms due to Bose-enhancement are identical up to $\vec{q} \leftrightarrow \vec{q}\,'$, but since $m_e \gg T$, the f_e change little under this exchange as well. We can therefore work with the simplified form

$$\mathcal{F} \approx f(\vec{p}\,')f_e(\vec{q}\,') - f(\vec{p})f_e(\vec{q})$$

Making use of these approximations and of the three-dimensional δ-function, we obtain

$$C[f(\vec{p})] = \frac{\pi}{4m_e^2|\vec{p}|}\int \frac{d^3q}{(2\pi)^3}\int \frac{d^3p'}{(2\pi)^3|\vec{p}\,'|}\,\delta\Big(|\vec{p}| + \frac{\vec{q}^{\,2}}{2m_e} - |\vec{p}\,'| - \frac{(\vec{q}+\vec{p}-\vec{p}\,')^2}{2m_e}\Big)$$

$$* \, |\mathcal{M}|^2 \left[f_e(\vec{q}+\vec{p}-\vec{p}\,') f(\vec{p}\,') - f_e(\vec{q}) f(\vec{p}) \right]$$

We further note that we may approximate for the energy transfer

$$E_e(\vec{q}) - E_e(\vec{q}+\vec{p}-\vec{p}\,') \approx \frac{\vec{q}^{\,2}}{2m_e} - \frac{(\vec{q}+\vec{p}-\vec{p}\,')^2}{2m_e} \underset{|\vec{q}| \gg |\vec{p}-\vec{p}\,'|,\ \text{typically}}{\approx} - \frac{\vec{q}\cdot(\vec{p}-\vec{p}\,')}{m_e}$$

We may therefore expand the δ-function as

$$\delta\left(|\vec{p}| + \frac{\vec{q}^{\,2}}{2m_e} - |\vec{p}\,'| - \frac{(\vec{q}+\vec{p}-\vec{p}\,')^2}{2m_e}\right) \approx \delta(|\vec{p}|-|\vec{p}\,'|) + \left(E_e(\vec{q}\,') - E_e(\vec{q})\right)$$

$$* \left. \frac{\partial \, \delta(|\vec{p}|-|\vec{p}\,'| + E(\vec{q}) - E(\vec{q}\,'))}{\partial E(\vec{q}\,')} \right|_{E_e(\vec{q}) = E_e(\vec{q}\,')}$$

$$= \delta(|\vec{p}|-|\vec{p}\,'|) + \frac{(\vec{p}-\vec{p}\,')\cdot\vec{q}}{m_e} \frac{\partial \delta(|\vec{p}|-|\vec{p}\,'|)}{\partial |\vec{p}\,'|}$$

When we moreover approximate $f_e(\vec{q}+\vec{p}-\vec{p}\,') \approx f_e(\vec{q})$ we obtain

$$C[f(\vec{p})] = \frac{\pi}{4 m_e^2 |\vec{p}|} \int \frac{d^3 q}{(2\pi)^3} f_e(\vec{q}) \int \frac{d^3 p'}{(2\pi)^3 |\vec{p}\,'|} |\mathcal{M}|^2$$

$$* \left[\delta(|\vec{p}|-|\vec{p}\,'|) + \frac{(\vec{p}-\vec{p}\,')\cdot\vec{q}}{m_e} \frac{\partial \delta(|\vec{p}|-|\vec{p}\,'|)}{\partial |\vec{p}\,'|} \right] \left(f(\vec{p}\,') - f(\vec{p}) \right)$$

In the non-relativistic limit, Compton scattering reduces to Thomson scattering with the squared matrix element

$$|\mathcal{M}|^2 = 8\pi \sigma_T m_e^2$$

where $\sigma_T = \frac{8\pi}{3} \frac{\alpha^2}{m_e^2}$ and $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$ is the Sommerfeld constant. If we were aiming for accuracies better than 1% (as the experimental data requires and as it is implemented in numerical codes), we should use the Klein-Nishima cross section. A full treatment is also required for the calculation of polarisations.

The first term in the collision integral is $n_e = \int \frac{d^3q}{(2\pi)^3} f_e(\vec{q})$, the number density of electrons. Moreover, we introduce $\vec{v}_b = \frac{\vec{q}}{m_e}$, the baryon velocity, following the astronomers' liberal usage of the term "baryon". Using the simplification of a constant $|\mathcal{M}|^2$, we arrive at

$$C[f(\vec{p})] = \frac{2\pi^2 n_e \sigma_T}{|\vec{p}|} \int \frac{d^3p'}{(2\pi)^3 |\vec{p}'|} \left[\delta(|\vec{p}|-|\vec{p}'|) + (\vec{p}-\vec{p}') \cdot \vec{v}_b \frac{\partial \delta(|\vec{p}|-|\vec{p}'|)}{\partial |\vec{p}'|} \right]$$

$$* \left[f^{(0)}(\vec{p}') - f^{(0)}(\vec{p}) - \frac{\partial f^{(0)}}{\partial |\vec{p}'|} |\vec{p}'| \Theta(\hat{p}') + \frac{\partial f^{(0)}}{\partial |\vec{p}|} |\vec{p}| \Theta(\hat{p}) \right]$$

$$= \frac{n_e \sigma_T}{4\pi |\vec{p}|} \int_0^\infty d|\vec{p}'| |\vec{p}'| \int d\Omega' \left[\delta(|\vec{p}|-|\vec{p}'|) \left(-\frac{\partial f^{(0)}}{\partial |\vec{p}'|} |\vec{p}'| \Theta(\hat{p}') + \frac{\partial f^{(0)}}{\partial |\vec{p}|} |\vec{p}| \Theta(\hat{p}) \right) \right.$$

$$\left. + (\vec{p}-\vec{p}') \cdot \vec{v}_b \frac{\partial \delta(|\vec{p}|-|\vec{p}'|)}{\partial |\vec{p}'|} \left(f^{(0)}(\vec{p}') - f^{(0)}(\vec{p}) \right) \right]$$

We have neglected here a contribution that is the product of a first order term with $\vec{v}_b$.

We introduce now the monopole part of the temperature perturbation:

$$\Theta_0(\vec{x}, t) = \frac{1}{4\pi} \int d\Omega' \, \Theta(\hat{p}', \vec{x}, t)$$

such that we can write

$$C[f(\vec{p})] = \frac{n_e \sigma_T}{|\vec{p}|} \int_0^\infty d|\vec{p}'| |\vec{p}'| \left[\delta(|\vec{p}|-|\vec{p}'|) \left(-\frac{\partial f^{(0)}}{\partial |\vec{p}'|} |\vec{p}'| \Theta_0 + \frac{\partial f^{(0)}}{\partial |\vec{p}|} |\vec{p}| \Theta(\hat{p}) \right) \right.$$

$$\left. + \vec{p} \cdot \vec{v}_b \frac{\partial \delta(|\vec{p}|-|\vec{p}'|)}{\partial |\vec{p}'|} \left(f^{(0)}(\vec{p}') - f^{(0)}(\vec{p}) \right) \right]$$

$$= -|\vec{p}| \frac{\partial f^{(0)}}{\partial |\vec{p}|} n_e \sigma_T \left[\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \vec{v}_b \right]$$

where we have integrated by parts.

From this form of the collision term we can already observe important properties of the baryon-photon plasma prior to recombination: Suppose that the Compton scattering is very efficient (tight coupling). Then all multipole moments of the temperature perturbation will decay, except for the monopole and the dipole, that is driven to be aligned with the bulk velocity of the baryons. This is an important simplification for solving for the evolution of Cosmic perturbations sufficiently long before recombination.

We can now combine the kinetic and the collision term to obtain

$$\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{d\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} = n_e \sigma_T \left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}_b \right]$$

We recall the definition $dt = a\, d\eta$ of the conformal time η and denote derivatives w.r.t. η with a prime:

$$\Theta' + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \Phi' + \hat{p}^i \frac{\partial \Psi}{\partial x^i} = n_e \sigma_T a \left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}_b \right]$$

This is a linear (in Θ, Φ, Ψ) partial differential equation. We can therefore perform a Fourier transformation that results in $\frac{\partial}{\partial x^i} \to i k_i$, such that we obtain ordinary differential equations separately for each $\vec{k}$-mode. The linear approximation works well before and during recombination and also up to about $z \sim 20$ afterwards. Numerical N-body simulations of the LSS in the non-linear regime use the linear evolution until $z \sim 20$ as an initial condition and then perform computations in real space.

We perform our Fourier transformations as

$$\Theta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \Theta(\vec{k}).$$

Note that this implies that the physical momentum is given by $\frac{\vec{k}}{a}$ whereas $\vec{k}$ is called the comoving momentum. The notation is a bit sloppy here as this implies that $\vec{k} = (-k_1, -k_2, -k_3)$ in terms of components of a four-vector. However, in the following, we refer to the components of $\vec{k}$ as k^i with an upper index and without minus sign and warn that this should not be a cause of confusion.

After these remarks, we define $k = \sqrt{\vec{k}^2} = \sqrt{k^i k^i}$ and the cosine of the angle between $\vec{k}$ and the photon direction $\hat{p}$ as $\mu = \frac{\vec{k}\cdot\hat{p}}{k}$. A photon aligned with the change in temperature has $|\mu| = 1$, whereas a photon travelling perpendicular has $\mu = 0$.

As a measure of the opacity of the plasma, we define the optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} d\eta' \, n_e \sigma_T a \implies \tau' = -n_e \sigma_T a$$

At late times, when n_e is small, $\tau \ll 1$, whereas $\tau \gg 1$ at early times.

Moreover, we assume that $\vec{\nabla} \times \vec{v}(\vec{x}) = 0$ such that

$$\hat{p}\cdot\vec{v}(\vec{x}) = \hat{p}\cdot\vec{\nabla} V(\vec{x}) \longrightarrow \hat{p}\cdot\underbrace{i\vec{k}\, V(\vec{k})}_{=\vec{v}(\vec{k})} = \frac{\hat{p}\cdot\vec{k}}{k}\, ik V(\vec{k}) = \mu\, v(\vec{k})$$

where we define $\hat{p}\cdot\vec{v}(\vec{k}) = \mu\, v(k)$.

With these definitions, we eventually obtain

$$\Theta' + ik\mu\Theta + \Phi' + ik\mu\Psi = -\tau'[\Theta_0 - \Theta + \mu v_b]$$

where Θ, Φ, Ψ and v_b are understood as functions of k.

2.3 The Boltzmann Equation for Cold Dark Matter (CDM)

By definition, this component is collisionless ("Dark") and non-relativistic ("Cold"). What needs to be done here is therefore to include mass into the kinetic term. The constraint on the four-momentum reads now

$$g_{\mu\nu} P^\mu P^\nu = m^2 = P^0 P^0(1+2\Psi) - a^2 P^i P^i (1+2\Phi) = P^0 P^0 (1+2\Psi) - p^2$$

where m is the mass of a Dark Matter particle and where as before $p^2 = -g_{ij} P^i P^j$. We also define $E = \sqrt{p^2+m^2}$ such that

$$P^\mu = \left(E(1-\Psi), p\hat{p}^i \frac{1-\Phi}{a}\right).$$

The total time derivative of the Dark Matter distribution f_{DM} can be written as

$$\frac{df_{DM}}{dt} = \frac{\partial f_{DM}}{\partial t} + \frac{\partial f_{DM}}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f_{DM}}{\partial E}\frac{dE}{dt} + \frac{\partial f_{DM}}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}$$

Again, the last term vanishes, being the product of two first order terms.

As for the other term, we can proceed by using the results for the photons and by applying the changes pertinent to the massive case:

$$\frac{dx^i}{dt} = \frac{P^i}{P^0} = \frac{p\hat{p}^i(1-\Phi)}{aE(1-\Psi)} = \frac{p}{E}\frac{\hat{p}^i}{a}(1+\Psi-\Phi)$$

$$\frac{dP^0}{d\lambda} = P^0\frac{dP^0}{dt} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta \Longrightarrow \frac{d}{dt}[E(1-\Psi)] = -\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{E}(1+\Psi)$$

$\Longrightarrow$

$$\frac{dE}{dt} = E\left(\frac{\partial\psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial\psi}{\partial x^i}\right) - \frac{p}{E}\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi)$$

$$= E\left(\cancel{\frac{\partial\psi}{\partial t}} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial\psi}{\partial x^i}\right)$$

$$-\frac{p}{E}(1-2\Psi)\left[\cancel{\frac{\partial\psi}{\partial t}\frac{E^2}{p}} + 2\frac{\partial\psi}{\partial x^i}\frac{E\hat{p}^i}{a} + p\left(\frac{\partial\Phi}{\partial t} + H\right)\right]$$

$$= -\frac{\partial\Psi}{\partial x^i}p\frac{\hat{p}^i}{a} - \frac{p}{E}p\left(H + \frac{\partial\Phi}{\partial t}\right)$$

We obtain, in summary,

$$\frac{df_{DM}}{dt} = \frac{\partial f_{DM}}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f_{DM}}{\partial x^i} - \frac{\partial f_{DM}}{\partial E}\left[\frac{p^2}{E}H + \frac{p^2}{E}\frac{\partial\Phi}{\partial t} + p\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] = 0$$

Now, we do not need to make any assumption about the form of f_{DM} (it may be of the Bose-Einstein, Fermi-Dirac or even non-equilibrium form), because the Dark Matter particles should be non-relativistic. We need however account for terms up to first order in $\frac{p}{m}$, in order to describe flow velocities. We accomplish this by taking moments of the Boltzmann equation. First, we integrate over $\frac{d^3p}{(2\pi)^3}$ and obtain:

$$\frac{\partial}{\partial t}\int\frac{d^3p}{(2\pi)^3}f_{DM} + \frac{1}{a}\frac{\partial}{\partial x^i}\int\frac{d^3p}{(2\pi)^3}f_{DM}\frac{p\hat{p}^i}{E} - \left[H + \frac{\partial\Phi}{\partial t}\right]\int\frac{d^3p}{(2\pi)^3}\frac{\partial f_{DM}}{\partial E}\frac{p^2}{E}$$

$$-\frac{1}{a}\frac{\partial\Psi}{\partial x^i}\int\frac{d^3p}{(2\pi)^3}\frac{\partial f_{DM}}{\partial E}\hat{p}^i p = 0$$

We define the Dark Matter density as

$$n_{DM} = \int\frac{d^3p}{(2\pi)^3}f_{DM}$$

and the velocity as

$$v^i = \frac{1}{n_{DM}}\int\frac{d^3p}{(2\pi)^3}f_{DM}\frac{p\hat{p}^i}{m} \simeq \frac{1}{n_{DM}}\int\frac{d^3p}{(2\pi)^3}f_{DM}\frac{p\hat{p}^i}{E}$$

Now, the first two terms can be expressed directly in terms of n_{DM} and v^i. The fourth term is second order in perturbations, and hence we neglect it. As for the third term, we write

$$\int \frac{d^3p}{(2\pi)^3} \frac{p^2}{E} \frac{\partial f_{DM}}{\partial E} = \int \frac{d^3p}{(2\pi)^3}\, p\, \frac{\partial f_{DM}}{\partial p} = \frac{1}{2\pi^2} \int_0^\infty p^3 dp\, \frac{\partial f_{DM}}{\partial p}$$

$$= -\frac{3}{2\pi^2} \int_0^\infty p^2 dp\, f_{DM} = -3\, n_{DM}$$

Hence, the zeroth moment of the Boltzmann equation leads to the continuity equation for Cold Dark Matter:

$$\frac{\partial n_{DM}}{\partial t} + \frac{1}{a} \frac{\partial (n_{DM} v^i)}{\partial x^i} + 3\left[H + \frac{\partial \Phi}{\partial t}\right] n_{DM} = 0$$

The zeroth order part reads

$$\frac{\partial n^{(0)}_{DM}}{\partial t} + 3H\, n^{(0)}_{DM} = 0 \iff \frac{1}{n^{(0)}_{DM}} \frac{\partial n^{(0)}_{DM}}{\partial t} = -\frac{3}{a} \frac{\partial a}{\partial t} \implies n^{(0)}_{DM} \propto a^{-3}$$

which is a consistent behaviour.

To extract the first order part, we define

$$n_{DM} = n^{(0)}_{DM}\left[1 + \delta(\vec{x}, t)\right] = n^{(0)}_{DM} + n^{(1)}_{DM} \quad \text{where } \delta = \frac{n^{(1)}_{DM}}{n^{(0)}_{DM}}$$

Note that since we assume CDM to be non-relativistic, $\delta = \frac{\delta\rho_{DM}}{\rho}$ is also the fractional energy-overdensity of Dark Matter. We obtain at first order:

$$\frac{\partial n^{(1)}_{DM}}{\partial t} + \frac{1}{a} \frac{\partial (n^{(0)}_{DM} v^i)}{\partial x^i} + 3H n^{(1)}_{DM} + 3 \frac{\partial \Phi}{\partial t} n^{(0)}_{DM} = 0$$

$$\implies \frac{\partial \delta}{\partial t} + \frac{1}{a} \frac{\partial v^i}{\partial x^i} + 3 \frac{\partial \Phi}{\partial t} = 0$$

$$\uparrow \; \frac{1}{n^{(0)}_{DM}} \frac{\partial n^{(1)}_{DM}}{\partial t} = \left(\frac{\partial}{\partial t} \delta\right) - n^{(1)}_{DM} \frac{\partial}{\partial t} \frac{1}{n^{(0)}_{DM}} = \left(\frac{\partial}{\partial t} \delta\right) + \frac{n^{(1)}_{DM}}{n^{(0)2}_{DM}}\, n^{(0)}_{DM} (-3H)$$

So far, we have introduced two perturbation variables,

δ and $\vec{v}$, but have only derived one equation. To obtain another, we take the first moment of the Boltzmann equation, i.e. we integrate over $\frac{d^3p}{(2\pi)^3}\frac{p}{E}\hat{p}^j$. This leads to

$$\frac{\partial}{\partial t}\int\frac{d^3p}{(2\pi)^3} f_{DM}\frac{p\hat{p}^j}{E} + \frac{1}{a}\frac{\partial}{\partial x^i}\int\frac{d^3p}{(2\pi)^3} f_{DM}\frac{p^2\hat{p}^i\hat{p}^j}{E^2} - \left[H+\frac{\partial\Phi}{\partial t}\right]\int\frac{d^3p}{(2\pi)^3}\frac{\partial f_{DM}}{\partial E}\frac{p^3\hat{p}^j}{E^2}$$

$$-\frac{1}{a}\frac{\partial\Psi}{\partial x^i}\int\frac{d^3p}{(2\pi)^3}\frac{\partial f_{DM}}{\partial E}\frac{\hat{p}^i\hat{p}^j p^2}{E} = 0$$

The second term is of order $\frac{p^2}{E^2}$ and can be neglected. To assess the third term, we note that

$$\int\frac{d^3p}{(2\pi)^3}\frac{\partial f_{DM}}{\partial E}\frac{p^3\hat{p}^j}{E^2} = \int\frac{d^3p}{(2\pi)^3}\frac{\partial f_{DM}}{\partial p}\frac{p^2\hat{p}^j}{E} \approx \int\frac{d\Omega\,\hat{p}^j}{(2\pi)^3}\int_0^\infty dp\,\frac{p^4}{E}\frac{\partial f_{DM}}{\partial p}$$

$$\frac{\partial p}{\partial E} = \frac{E}{p}$$

$$\frac{\partial}{\partial p}\frac{1}{E} = -\frac{1}{E^2}\frac{p}{E}$$

$$= -\int\frac{d\Omega\,\hat{p}^j}{(2\pi)^3}\int_0^\infty dp\, f_{DM}\left(\frac{4p^3}{E} - \frac{p^5}{E^3}\right) = -4\, n_{DM}\, v^j$$

where we have neglected the second term in the round brackets.

In order to evaluate the fourth term, we note that

$\int d\Omega\, \hat{p}^i\hat{p}^j = A\,\delta^{ij}$ with A to be determined

$$\overset{\mathrm{tr}}{\Longrightarrow} \int d\Omega = 4\pi = 3A \Leftrightarrow A = \frac{4\pi}{3} \Longrightarrow \int d\Omega\,\hat{p}^i\hat{p}^j = \frac{4\pi}{3}\delta^{ij}$$

We can therefore evaluate and approximate

$$\int\frac{d^3p}{(2\pi)^3}\frac{\partial f_{DM}}{\partial E}\frac{\hat{p}^i\hat{p}^j p^2}{E} \approx \frac{1}{(2\pi)^3}\int d\Omega\,\hat{p}^i\hat{p}^j\int_0^\infty dp\,\frac{\partial f_{DM}}{\partial p}p^3$$

$$= -\frac{1}{(2\pi)^3}\frac{4\pi}{3}\delta^{ij}\int_0^\infty dp\,3p^2 f_{DM} = -\delta^{ij}\, n_{DM}$$

In total, we obtain for the first moment of the Boltzmann equation have neglected contribution $\propto \Phi$ here as higher order

$$\frac{\partial(n_{DM} v^j)}{\partial t} + 4H n_{DM} v^j + \frac{1}{a} n_{DM} \frac{\partial \Psi}{\partial x^j} = 0 \qquad \text{recall: } \frac{\partial n_{DM}^{(0)}}{\partial t} = -3H n_{DM}^{(0)}$$

$$\Longrightarrow \frac{\partial v^j}{\partial t} + H v^j + \frac{1}{a}\frac{\partial \Psi}{\partial x^j} = 0$$

This equation together with the differential equation for δ governs the evolution of the density and velocity of CDM.
Note that the lth moment of a Boltzmann equation depends on the lth and $l+1$th moments of the distribution: the zeroth moment equation depends on δ and v. The first moment equation depends here on v only, because we have truncated higher moments based on our assumption $p \ll E$ (CDM is non-relativistic). So it is due to this approximation that we obtain here a closed set of fluid equations.
We assume that the velocity is irrotational, such that we may write $\tilde{v}^i = \frac{k^i}{k} v$. In Fourier space, the fluid equations then read

$$\tilde{\delta}' + ik\tilde{v} + 3\tilde{\Phi}' = 0$$

$$\tilde{v}' + \frac{a'}{a}\tilde{v} + ik\tilde{\Psi}' = 0$$

2.4 The Boltzmann Equation for Baryons

Astronomers believe that electrons are baryons. For CMB calculations however, it makes indeed sense to group these together, as these are coupled through Coulomb scattering $e + p \longrightarrow e + p$. Since the rate for Coulomb scattering is much larger than the Hubble rate, e and p are tightly coupled and forced to assume a common over=

density, described by the baryonic density contrast

$$\frac{\rho_e - \rho_e^{(0)}}{\rho_e} = \frac{\rho_p - \rho_p^{(0)}}{\rho_p^{(0)}} = \delta_b$$

Note of course that tight coupling ends at recombination, where neutral hydrogen is formed.

Tight coupling also holds for the velocities

$\vec{v}_e = \vec{v}_p = \vec{v}_b$

To derive equations for δ_b and $\vec{v}_b$, we start from

$$\frac{df_e(\vec{x},\vec{q},t)}{dt} = \langle C_{ep} \rangle_{QQ'q'} + \langle C_{e\gamma} \rangle_{pp'q'}$$

$$\frac{df_p(\vec{x},\vec{Q},t)}{dt} = \langle C_{ep} \rangle_{qq'Q'}$$

The Klein-Nishina cross section is $\propto \frac{1}{m_{e,p}^2}$, such that we may neglect here the Compton scattering of protons.

Concerning the notation, we make use of the following conventions: The photon momenta are $\vec{p}$ and $\vec{p}'$, the electron momenta $\vec{q}$ and $\vec{q}'$ and the proton momenta $\vec{Q}$ and $\vec{Q}'$.

Now consider for example the Compton scattering term in the equation for the electron distribution function:

$$C_{e\gamma} = (2\pi)^4 \delta^4(p+q-p'-q') \frac{|\mathcal{M}|^2}{8E(p)E(p')E_e(q)E_e(q')} \left(f_e(q') f_\gamma(p') - f_e(q) f_\gamma(p) \right)$$

The angular brackets then denote integration over all momenta in the subscripts:

$$\langle \ldots \rangle_{pp'q'} = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} (\ldots)$$

The Coulomb collision terms are to be understood accordingly.

Integrating both sides of the equation for the electron distribution over $\frac{d^3q}{(2\pi)^3}$ (i.e. taking the first moment), we can use the result worked out for CDM to obtain

$$\frac{\partial n_e}{\partial t} + \frac{1}{a}\frac{\partial(n_e v_b^i)}{\partial x^i} + 3\left[H + \frac{\partial \Phi}{\partial t}\right] n_e = \langle c_{ep}\rangle_{QQ'q'q} + \langle c_{e\gamma}\rangle_{pp'q'q}$$

Both of the collision terms on the right vanish, because the integration measure is symmetric under the exchange $Q \longleftrightarrow Q'$, $q \longleftrightarrow q'$, while the integrand is antisymmetric. This should be the case, because the interactions we consider conserve electron number. The perturbed version of this equation therefore takes the same form as for CDM,

$$\tilde{\delta}_b' + ik\tilde{v}_b + 3\tilde{\Phi}' = 0$$

Now for the first moment equations, we multiply these with $\vec{q}$ ($\vec{Q}$) for the electrons (protons), instead of $\frac{\vec{q}}{E}$ in the case of CDM. Since $E \simeq m_{e,p}$ in the non-relativistic approximation, the left hand side of the equations are simply multiplied with $m_{e,p}$ compared to the CDM case. When we add both equations, the proton terms dominate on the left and we obtain

$$m_p \frac{\partial(n_b v_b^j)}{\partial t} + 4H\, m_p n_b v_b^j + \frac{m_p n_b}{a}\frac{\partial \Psi}{\partial x^i}$$

$$= \langle c_{ep}(q^j + Q^j)\rangle_{QQ'q'q} + \langle c_{e\gamma}\, q^j\rangle_{pp'q'q}$$

The integrated Coulomb collision term (i.e. the first term on the right-hand-side) vanishes because these scatterings conserve the total momentum carried by electrons and protons. This is different for the second term, because

momentum is exchanged with the photons.
We denote the zeroth-order baryon density with $\varrho_b = n_b^{(0)} m_p$ (such that the total density is $\varrho_b(1+\delta_b)$).
Dividing both sides by ϱ_b, we obtain

$$\frac{\partial v_b^j}{\partial t} + H v_b^j + \frac{1}{a}\frac{\partial \psi}{\partial x^j} = \frac{1}{\varrho_b}\langle C_{e\gamma}\, q^j \rangle_{pp'q'q}$$

To evaluate the integral over the Compton scattering term, note that

$$\langle C_{e\gamma}\vec{q}\rangle_{pp'q'q} = -\langle C_{e\gamma}\vec{p}\rangle_{pp'q'q},$$

due to the total momentum conservation within electrons and photons. We also recall the definition $\mu = \frac{\vec{k}\cdot\hat{p}}{|\vec{k}|}$ as the cosine of the angle between the wavenumber $\vec{k}$ and the photon momentum $\vec{p}$.
We go to Fourier space and multiply both sides of the first moment equation with $\hat{k}^j$. We also recall the approximate form of the Compton scattering term from Section 2.2. We thus obtain (writing $p \equiv |\vec{p}|$)

$$-\frac{\langle C_{e\gamma}\, p\mu\rangle_{pp'q'q}}{\varrho_b} = \frac{n_e \sigma_T}{\varrho_b}\int \frac{d^3p}{(2\pi)^3}\, p^2 \frac{\partial f^{(0)}}{\partial p}\,\mu\left[\tilde{\Theta}_0 - \tilde{\Theta}(\mu) + \tilde{v}_b\,\mu\right]$$

$$= \frac{n_e\sigma_T}{\varrho_b}\int_0^\infty \frac{dp}{4\pi^2}\, p^4 \frac{\partial f^{(0)}}{\partial p}\int_{-1}^{1} d\mu\,\mu\left[\tilde{\Theta}_0 - \tilde{\Theta}(\mu) + \tilde{v}_b\,\mu\right]$$

It makes sense now to generalise the definition of the temperature monopole and to introduce

$\Theta_1 = i\int_{-1}^{1}\frac{d\mu}{2}\,\mu\,\Theta(\mu)$, where the factor $\frac{i}{2}$ is a convention.

Putting everything together, we obtain the equation

$$\tilde{v}_b' + \frac{a'}{a}\tilde{v}_b + ik\tilde{\Psi} = \tau' \frac{4\varrho_\gamma}{3\varrho_b}\left[3i\,\tilde{\Theta}_1 + \tilde{v}_b\right]$$

We remark that also neutral hydrogen and helium are effectively tightly coupled to electrons and protons. Above equation therefore describes the baryon velocity even at times after recombination.

2.5 Summary and Generalisations

Our coupled equations in Fourier space solve for the scalar quantities $\delta(k,\eta)$, $v(k,\eta)$, $\delta_b(k,\eta)$, $v_b(k,\eta)$, $\Theta(k,\mu,\eta)$, where we drop now the tilde. The velocities are scalar in the sense that these are parallel to k. Moreover, the evolution equations for CDM and baryons depend on k only, such that these are isotropic, whereas the temperature perturbation also depends on the direction μ.

We extend our definition of the temperature monopole and dipole to the multipole

$$\Theta_\ell = \frac{1}{(-i)^\ell} \int_{-1}^{1} \frac{d\mu}{2} P_\ell(\mu)\, \Theta(\mu)$$

where P_ℓ is a Legendre polynomial. We can therefore describe the temperature perturbation either in terms of $\Theta(k,\mu,\eta)$ or the $\Theta_\ell(k,\eta)$.

To our network of equations, we add photon polarisation Θ_P, that we discuss in more detail at a later point.

We also add an equation for the perturbation in the neutrino distribution $\mathcal{N}$, the form of which can be inferred from the equation for the temperature perturbation, because both, photons and neutrinos are relativistic.

Moreover, we have neglected so far the the angular dependence of Compton scattering, which brings up terms

involving the second Legendre polynomial $P_2(\mu) = \frac{3\mu^2 - 1}{2}$.
In summary, we have the following network of Boltz= mann equations:

$$\Theta' + ik\mu\Theta = -\Phi' - ik\mu\Psi - \tau'\left[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

where $\Pi = \Theta_2 + \Theta_{P2} + \Theta_{P0}$

$$\Theta_P' + ik\mu\Theta_P = -\tau'\left[-\Theta_P + \frac{1}{2}(1 - P_2(\mu))\Pi\right]$$

$$\delta' + ikv = -3\Phi'$$

$$v' + \frac{a'}{a}v = -ik\Psi$$

$$\delta_b' + ikv_b = -3\Phi'$$

$$v_b' + \frac{a'}{a}v_b = -ik\Psi + \frac{\tau'}{R}\left[v_b + 3i\Theta_1\right]$$ where $\frac{1}{R} = \frac{4\rho_\gamma^{(0)}}{3\rho_b^{(0)}}$

$$\mathcal{N}' + ik\mu\mathcal{N} = -\Phi' - ik\mu\Psi$$