# 1. The Homogeneous Universe

## 1.1 General Relativity

In General Relativity, the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

is invariant under differentiable coordinate transformations.
The theory reduces to Special Relativity when we use $g_{\mu\nu} = \eta_{\mu\nu}$, where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and when we restrict the coordinate transformations to Lorentz transformations.
Apparently, an expanding Universe can be described through the Friedmann-Robertson-Walker (FRW) metric:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix}.$$

It describes a homogeneous, expanding Universe, where the physical distance between two galaxies (without peculiar velocities) grows $\propto a(t)$.
The motion of test particles is described by the geodesic equation. To derive it, we first consider Newton-Galilei mechanics in a two-dimensional plane.
In Cartesian coordinates, the equations of motion (EOMs) are given by $\dfrac{d^2 x^i}{dt} = 0$.

Now, if we use polar coordinates $x'^i = (r, \vartheta)$, where $x^1 = r\cos\vartheta = x'^1 \cos x'^2$ and $x^2 = r \sin\vartheta = x'^1 \sin x'^2$. The EOMs then take a different form, that is obtained from transforming the Cartesian equations. For this purpose, we use

$$\frac{dx^i}{dt} = \frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt} \quad \text{where} \quad \frac{dx}{dx'} = \begin{pmatrix} \cos x'^2 & -x'^1 \sin x'^2 \\ \sin x'^2 & x'^1 \cos x'^2 \end{pmatrix}$$

is called the transformation matrix. We thus obtain that

$$\frac{d}{dt}\frac{dx^i}{dt} = \frac{d}{dt}\frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt} = 0$$

To derive a form, where the time derivatives act on $x'$ only, we note that

$$\frac{d}{dt}\frac{\partial x^i}{\partial x'^j} = \frac{\partial}{\partial x'^j}\frac{dx^i}{dt} = \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\frac{dx'^k}{dt}$$

It follows

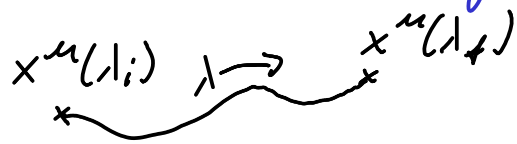
$$\frac{d}{dt}\frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt} = \frac{\partial x^i}{\partial x'^j}\frac{d^2 x'^j}{dt^2} + \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\frac{dx'^k}{dt}\frac{dx'^j}{dt} = 0$$

We eventually multiply with the inverse of the transformation matrix from the left and obtain the form

$$\left[\left(\frac{\partial x}{\partial x'}\right)^{-1}\right]^l{}_i * \ldots \Longrightarrow \frac{d^2 x'^l}{dt^2} + \underbrace{\left(\left[\left(\frac{\partial x}{\partial x'}\right)^{-1}\right]^l{}_i \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\right)}_{=:\Gamma^l_{jk}}\frac{dx'^k}{dt}\frac{dx'^j}{dt} = 0$$

This defines the Christoffel symbol $\Gamma$, that apparently vanishes in Cartesian coordinates. In curved spacetimes (in particular also for the FRW metric) it is not possible in general to find coordinates where $\Gamma$ vanishes.

To obtain a geodesic equation for Relativity, we must account for the fact that time is now a coordinate. In the geodesic equation, we must replace it with a parameter $\lambda$ that monotonically increases along the particle path:




The geodesic equation then reads:

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

Since there should be a <u>local</u> Minkowski system where

$ds^2 = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$ i.e. $x \leftrightarrow x'$ to be replaced in $\xi \leftrightarrow x$ above example

we may express

$$g_{\mu\nu}(x) = \eta_{\alpha\beta}\frac{d\xi^\alpha}{dx^\mu}\frac{d\xi^\beta}{dx^\nu} \quad \text{and} \quad \Gamma^\varkappa_{\mu\nu} = \frac{\partial x^\varkappa}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}$$

We can therefore express the Christoffel symbols in terms of first derivatives of the metric tensor, and a straightforward calculation then leads to

$$\Gamma^\varkappa_{\lambda\mu} = \frac{g^{\varkappa\nu}}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right)$$

Before we go on calculating these expressions for the FRW metric, we remind of the fact that the Christoffel Symbols are no tensors. They are of great practical use however in constructing the covariant derivative $\nabla_\mu$ (a first-rank) tensor through

$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} = \partial_\sigma T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} &+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\cdots}{}_{\nu_1\nu_2\cdots} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\cdots}{}_{\nu_1\nu_2\cdots} \\ &- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\cdots}{}_{\lambda\nu_2\cdots} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2\cdots}{}_{\nu_1\lambda\cdots}\end{aligned}$$

Another useful consequence of this construction is that the $\Gamma$ are symmetric in their lower indices (i.e. they are torsion free) and that

$$\nabla_\rho g_{\mu\nu} = 0 , \quad \nabla_\rho g^{\mu\nu} = 0 ,$$

such that raising and lowering of indices commutes with the covariant derivative.

Now for the case of the FRW metric. From its special form, we obtain ($x^0 = t$)

$$\Gamma^0_{\alpha\beta} = \frac{1}{2}\left[\frac{\partial g_{\alpha 0}}{\partial x^\beta} + \frac{\partial g_{\beta 0}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^0}\right] = -\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^0}$$

$$\Longrightarrow \quad \Gamma^0_{00} = 0$$

$$\Gamma^0_{0i} = \Gamma^0_{i0} = 0$$

$$\Gamma^0_{ij} = \delta_{ij}\, a\dot{a}$$

$$\Gamma^i_{0j} = \Gamma^j_{0i} = \delta_{ij}\frac{\dot{a}}{a} \quad \text{and} \quad \Gamma^i_{\alpha\beta} = 0 \ \text{otherwise.}$$

We now relate these results to the notion of redshift in the expanding Universe. Consider a four-momentum vector

$$p^\alpha = \frac{dx^\alpha}{d\lambda} \quad \text{and} \quad p^\alpha = (E, \vec{p}).$$

For a massive particle, we can identify $d\lambda = \frac{d\tau}{m}$, where $m$ is the mass and $\tau$ the proper time. More generally, we may write

$$\frac{d}{d\lambda} = \frac{dx^0}{d\lambda}\frac{d}{dx^0} = E\frac{d}{dt}$$

The 0-component of the geodesic equation then reads

$$\frac{d^2x^0}{d\lambda^2} = E\frac{d}{dt}E = -\Gamma^0_{ij}\, p^i p^j = -\delta_{ij}\, p^i p^j a\dot{a}$$

A massless particle satisfies $p^2 = g_{\mu\nu} p^\mu p^\nu = E^2 - a^2 \vec{p}^{\,2} = 0$

$$\Rightarrow E \frac{d}{dt} E = -E^2 \frac{\dot a}{a} \Rightarrow \frac{dE}{E} = -\frac{da}{a} \Rightarrow E \propto \frac{1}{a}$$

in agreement with the expected redshift property of photons.

So far, we have considered test particles in a given background geometry. The latter is determined by the Einstein equations

$$\underbrace{G_{\mu\nu}}_{\text{Einstein tensor}} := \underbrace{R_{\mu\nu}}_{\text{Ricci tensor}} - \frac{1}{2} g_{\mu\nu} \underbrace{R}_{\text{Ricci (curvature) scalar}} = 8\pi \underbrace{G}_{\text{Newton's constant}} \underbrace{T_{\mu\nu}}_{\text{energy-momentum tensor}}$$

The Ricci tensor can be expressed as

$$R_{\mu\nu} = \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} - \frac{\partial \Gamma^\alpha_{\mu\alpha}}{\partial x^\nu} + \Gamma^\alpha_{\beta\alpha} \Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu} \Gamma^\beta_{\mu\alpha}$$

and the scalar curvature $R = R^\mu{}_\mu$.

Using above results, we obtain

$$R_{00} = \underbrace{\frac{\partial \Gamma^\alpha_{00}}{\partial x^\alpha}}_{=0} - \frac{\partial \Gamma^\alpha_{0\alpha}}{\partial x^0} + \underbrace{\Gamma^\alpha_{\beta\alpha} \Gamma^\beta_{00}}_{=0} - \Gamma^\alpha_{\beta 0} \Gamma^\beta_{0\alpha}$$

$$= -3 \frac{\partial}{\partial t} \frac{\dot a}{a} - 3 \left(\frac{\dot a}{a}\right)^2 = -3 \frac{\ddot a}{a}$$

$$R_{ij} = \frac{\partial \Gamma^\alpha_{ij}}{\partial x^\alpha} - \underbrace{\frac{\partial \Gamma^\alpha_{i\alpha}}{\partial x^j}}_{=0} + \underbrace{\Gamma^\alpha_{\beta\alpha}}_{\substack{\beta=0 \\ \to \neq 0 \\ \text{iff. } \alpha=1,2,3}} \underbrace{\Gamma^\beta_{ij}}_{\substack{\neq 0 \text{ iff. } \\ \beta=0}} - \underbrace{\Gamma^\alpha_{\beta j} \Gamma^\beta_{i\alpha}}_{\substack{\neq 0 \text{ iff. either} \\ \alpha \text{ or } \beta=0}}$$

$$= \delta_{ij} \frac{\partial}{\partial t} a \dot a + \delta_{ij} a \dot a \, 3 \frac{\dot a}{a} - \delta_{ij} \, 2 \, a \dot a \frac{\dot a}{a}$$

$$= \delta_{ij} \left( 2 \dot a^2 + a \ddot a \right)$$

$$R = R^{\mu}{}_{\mu} = 3\frac{\ddot{a}}{a} - \frac{1}{a^2}\left(6\dot{a}^2 + 3a\ddot{a}\right) = -6\frac{\ddot{a}}{a} - 6\frac{\dot{a}^2}{a^2}$$

A stress-energy tensor that agrees with the symmetries of the metric is given by

$T^{\mu}{}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p)$ or $T^0{}_0 = \rho$ and $T^i{}_i = -p\, g^i{}_i$

where $\rho(t)$ is the energy density and $p(t)$ is the pressure density.

In all generality, the FRW metric takes the form

$$ds^2 = dt^2 - a^2(t)\left[\frac{dx^2}{1-Kx^2} + x^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right]$$

For $K=0$, we can recover the metric that we have been discussing so far, whereas $K>0$ ($K<0$) corresponds to a spatially closed (open) model. Since the observed Universe is close to $K=0$ (spatially flat), we enjoy a substantial simplification throughout most of our calculations. For $K \neq 0$, above results generalise to

$$R_{00} = -3\frac{\ddot{a}}{a}, \qquad R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2}\right] g_{ij}$$

$$R = -6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right].$$

Now, consider the $00$ component of the Einstein equations:

$$R_{00} - \frac{1}{2} g_{00} R = 8\pi G T_{00} \Longrightarrow$$

$$-3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} = 8\pi G \rho \Longrightarrow$$

$$\boxed{\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi}{3} G \rho}$$ Friedmann equation

The ii component yields

$$-\frac{\ddot{a}}{a}-2\frac{\dot{a}^2}{a^2}-\frac{2k}{a^2}+3\frac{\ddot{a}}{a}+3\frac{\dot{a}^2}{a^2}+\frac{3k}{a^2}=-8\pi G p$$

$$\Rightarrow 2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}=-8\pi G p$$

We can eliminate the terms $\propto \frac{\dot{a}^2}{a^2}$ to obtain another useful form,

$$\frac{\ddot{a}}{a}=-\frac{4\pi}{3}G(\rho+3p)$$

that allows to determine the acceleration or deceleration of the Universe.

A very useful quantity is the Hubble parameter $H=\frac{\dot{a}}{a}$.

In terms of this, we recast the Friedmann equation as

$$\frac{k}{H^2a^2}=\frac{\rho}{\frac{3H^2}{8\pi G}}-1=\frac{\rho}{\rho_c}-1=\Omega-1,$$

where we have introduced the critical density $\rho_c$. The observation, that the spatial curvature of the Universe must be very small can therefore alternatively be stated as its energy density being clos to critical.

## 1.2 Distance Measures

The redshift $z$ offers itself as a practical method to determine the distance of objects in the Universe. It is awkwardly defined as

$$1+z=\frac{\lambda_0}{\lambda}=\frac{a_0}{a},$$

where $\lambda_0$ is the wavelength that is observed on Earth today and $a_0$ is the scale factor today,

the value of which is a matter of definition and may conveniently be set to one for many practical purposes. The natural wavelength observed by the emitter is $\lambda$. We note that above relation assumes the absence of peculiar velocities that are a notorious source of uncertainty in distance measurements.
Another common distance measure is the comoving distance between two points $x_1$ and $x_2$

$$\chi = a_0^2(\vec{x}_1 - \vec{x}_2)^2 = a_0^2 \sum_i (x_1^i - x_2^i)^2$$

For particles at rest, $\dot{x}_{1,2}^i = 0$, such that the comoving distance is conserved as the Universe expands. It corresponds to the physical spatial distance today.
It is often convenient to perform a coordinate transformation $dt = a\,d\eta$ such that $g_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu}$, where we call $\eta$ the <u>conformal time.</u>
Because $a^2(d\vec{x})^2 = dt^2$, within a time interval $dt$, light travels the distance $\sqrt{(d\vec{x})^2} = \frac{dt}{a} = d\eta$. By a certain choice of integration constant (that may not always be the most convenient one), we define $\eta$ as an interesting quantity, the <u>comoving horizon:</u>

$$\eta = \int_0^t \frac{dt'}{a(t')}$$

This is the comoving distance that light has travelled from the Big Bang at $t=0$ to an observer at the time $t$.
This concept can be generalised to light emitted at a certain scale factor $a$:

$$\chi(a) = \int_{t(a)}^{t_0} \frac{dt'}{a(t')} \underset{\frac{da}{dt} = aH}{=} \int_{a}^{a_0} da' \, \frac{1}{a'^2 H(a')}$$

This corresponds to above definition of $\chi$ for $\vec{x}_1 = \vec{0}$ and $|\vec{x}_2| = \chi(a)$.

Another measure is the <u>angular diameter distance</u>

$$d_A = \frac{\ell}{\vartheta},$$

where $\ell$ is the physical size of an object and $\vartheta$ the observed subtended angle.

In a spatially flat Universe, the comoving size of the observed object is $\frac{\ell}{a}$, so $\vartheta = \frac{\ell}{a\chi(a)}$, where $\chi(a)$ is the comoving distance from the observer. We can conclude that

$$d_A^{flat} = a\chi = \frac{\chi}{1+z}$$

For a non-flat Universe, we define the curvature density $\Omega_k = 1 - \Omega_0$, and above expression generalises to

$$d_A = \frac{a}{H_0\sqrt{|\Omega_k|}} \begin{cases} \sinh\left(\sqrt{\Omega_k}\, H_0 \chi\right) & \text{for } \Omega_k > 0 \\ \sin\left(\sqrt{-\Omega_k}\, H_0 \chi\right) & \text{for } \Omega_k < 0 \end{cases}$$

Another important distance measure is the <u>luminosity distance</u>. In a flat, non-expanding space, the observed flux $F$ from a source with luminosity $L$ in a distance $d$ from the source is $F = \frac{L}{4\pi d^2}$.

Due to the time dilation and the red-shift, the corresponding formula for a spatially flat Friedmann Universe is $F = \frac{a^2 L}{4\pi \chi^2(a)}$. Hence, the formula for flat space is still valid when replacing $d$ with the luminosity distance $d_L = \frac{\chi}{a}$.

## 1.3 Unperturbed Cosmic Fluids

We recall that General Relativity can be formulated field theoretically through the Einstein-Hilbert action

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x \sqrt{-g}\,(R + 2\Lambda)$$

and the "matter" action

$$S_M = \int d^4x \sqrt{-g}\,\mathcal{L}_M$$

which combine to the total action as $S = S_{EH} + S_M$. Variation with respect to $g_{\mu\nu}$ yields:

$$\delta S_{EH} = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}R g^{\mu\nu} - \Lambda g^{\mu\nu}\right)\delta g_{\mu\nu}$$

$$\delta S_M = -\frac{1}{2}\int d^4x \sqrt{-g}\, T^{\mu\nu}\,\delta g_{\mu\nu}$$

Apparently, imposing that $\delta S = 0$ for general small variations implies the Einstein equations. We have also introduced here the cosmological constant $\Lambda$, that can be absorbed into the stress-energy tensor as

$$T^\mu{}_\nu + \frac{1}{8\pi G}\Lambda\,\delta^\mu{}_\nu \longmapsto T^\mu{}_\nu$$

We note that the $\mu = 0$ component of stress-energy conservation $\nabla_\nu T^{\mu\nu} = 0$ leaves us with

$$\partial_\nu T^{0\nu} + \Gamma^0_{\nu\lambda}T^{\lambda\nu} + \Gamma^\nu_{\nu\lambda}T^{0\lambda} = \partial_0 T^{00} + a\dot{a}\,T^{ii} + 3\frac{\dot{a}}{a}T^{00}$$

$$= \partial_t \rho + \frac{\dot{a}}{a}3p + 3\frac{\dot{a}}{a}\rho = 0$$

$\Longrightarrow$

$dE = -p\,dV$

$$d(a^3\rho) = -p\, d a^3 \iff d[a^3(\rho+p)] = a^3\, dp$$

This is the first law of thermodynamics in an expanding volume $\propto a^3$.

Now, consider a fluid that may be described by the equation of state $p = w\rho$. Recall that for $w < -\frac{1}{3}$, the Universe accelerates, whereas for $w > -\frac{1}{3}$, it decelerates. We immediately deduce that

$$d(a^3\rho) = 3a^2\rho\, da + a^3 d\rho = -w\rho\, 3a^2 da$$

$$\Longrightarrow \frac{d\rho}{\rho} = -3(1+w)\frac{da}{a} \Longrightarrow \rho \propto a^{-3(1+w)}$$

Basic Fluids of Cosmological interest are:

Radiation:

Relativistic radiation (cf. statistical mechanics) has the equation of state $p = \frac{1}{3}\rho \iff w = \frac{1}{3} \Longrightarrow \rho \propto a^{-4}$

Matter:

Non-relativistic matter behaves like pressureless dust, hence: $p = 0 \iff w = 0 \Longrightarrow \rho \propto a^{-3}$

Vacuum Energy:

$p = -\rho \iff w = -1 \Longrightarrow \rho = \text{const.}$

A cosmological constant behaves like vacuum energy with $\rho = \frac{1}{8\pi G}\Lambda$

Curvature:

Recall the Friedmann Equation $H^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2}$.

In some sense, one may interpret curvature as a fluid scaling $\sim a^{-2}$.

Now, according to the Planck collaboration,

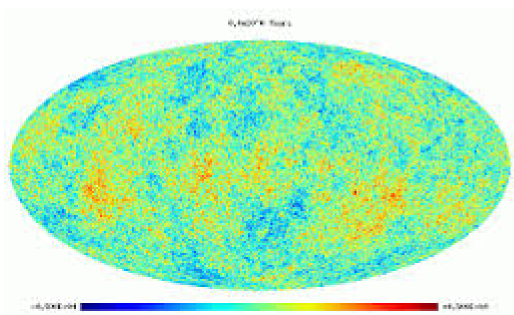
$$\Omega_\Lambda = 0{,}685^{+0{,}018}_{-0{,}016} \quad (68\% \text{ c.l. } 2013)$$

$$\Omega_m = 0{,}315^{+0{,}016}_{-0{,}018}$$

$$|\Omega_k| \lesssim 1\% \quad (95\% \text{ c.l.})$$

$$z_{eq} = 3391 \pm 30 \quad \text{(red shift of matter-radiation equality)}$$

Perhaps the main goal of these lectures is to explain how these and other numbers are determined to such astonishing precision. To this end, we present the iconic plots that illustrate theory & observations:



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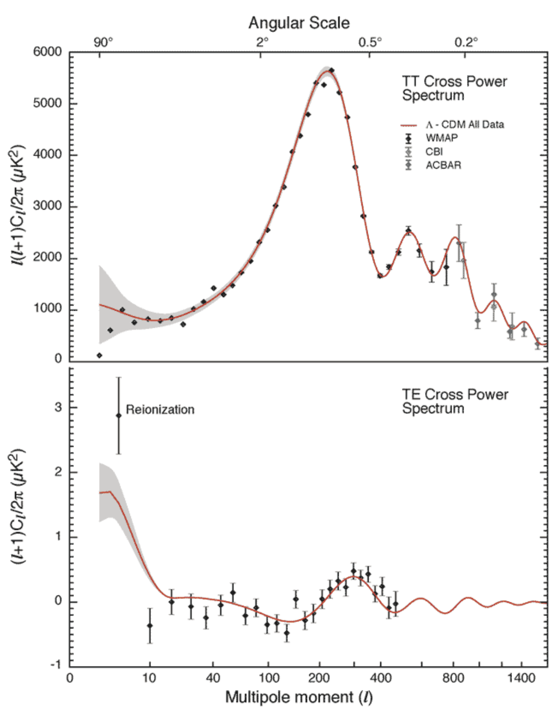
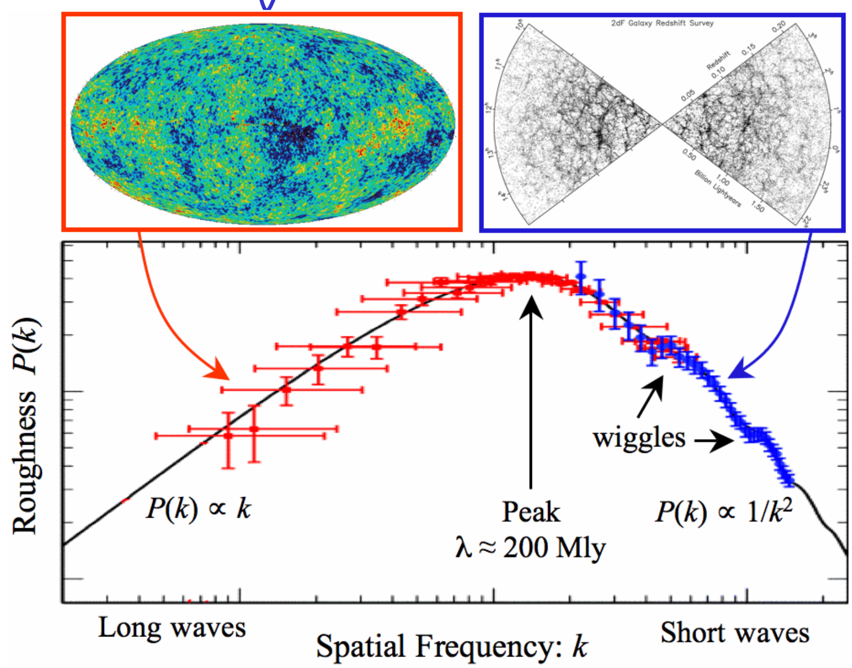
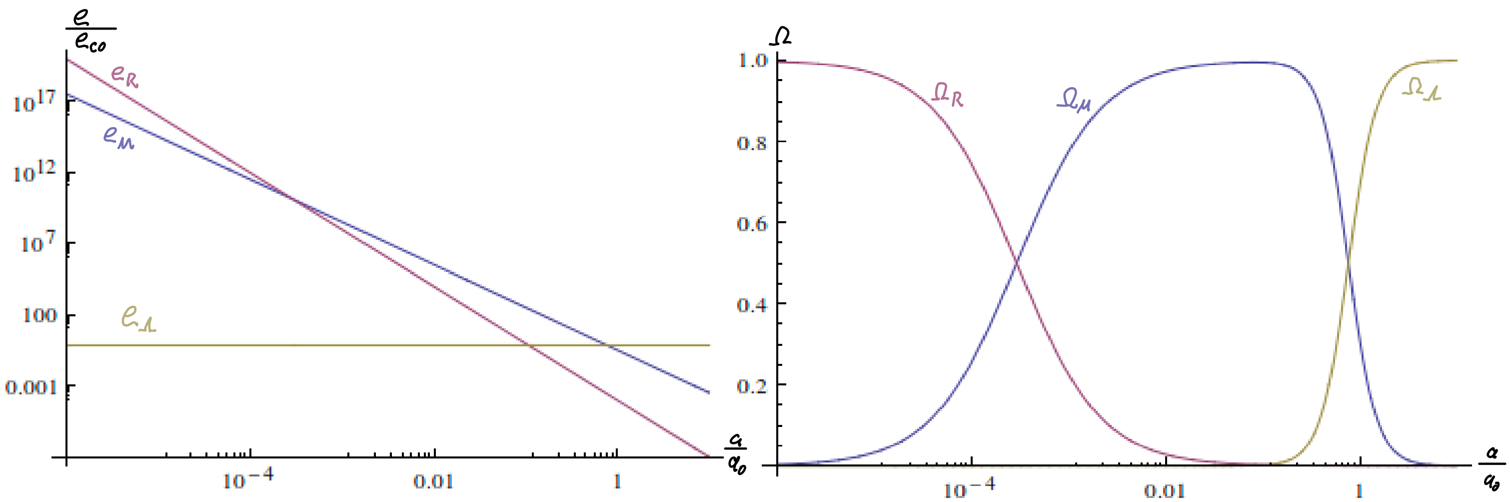



Above numbers already tell a big deal about the cosmic history. We basically see that the energy density of the Universe underwent epochs of radiation and of matter domination, while presently, we appear to be in the transition to a $\Lambda$-dominated epoch.




Note that provided $\Omega_k$ is at the sub-percent level today, it must have been even more closely to zero earlier on, a tuning that requires explanation. This is known as the <u>flatness problem.</u>

It is also amusing to note that provided the present cosmic acceleration continues, all galaxies except for Andromeda will disappear beyond the observable horizon within a few billion years. The future observer therefore will never discover the expansion of the Universe, leave alone acceleration due to $\Lambda$.

It is a good point now to get back again to the distance measures. A Universe with a cosmological constant had a smaller energy density at early times and was consequently expanding more slowly than a Universe without $\Lambda$. For a given red-shift, light had therefore more time to travel to us, such that $\chi(z)$, $d_A(z)$ and $d_L(z)$ are larger than in a Universe

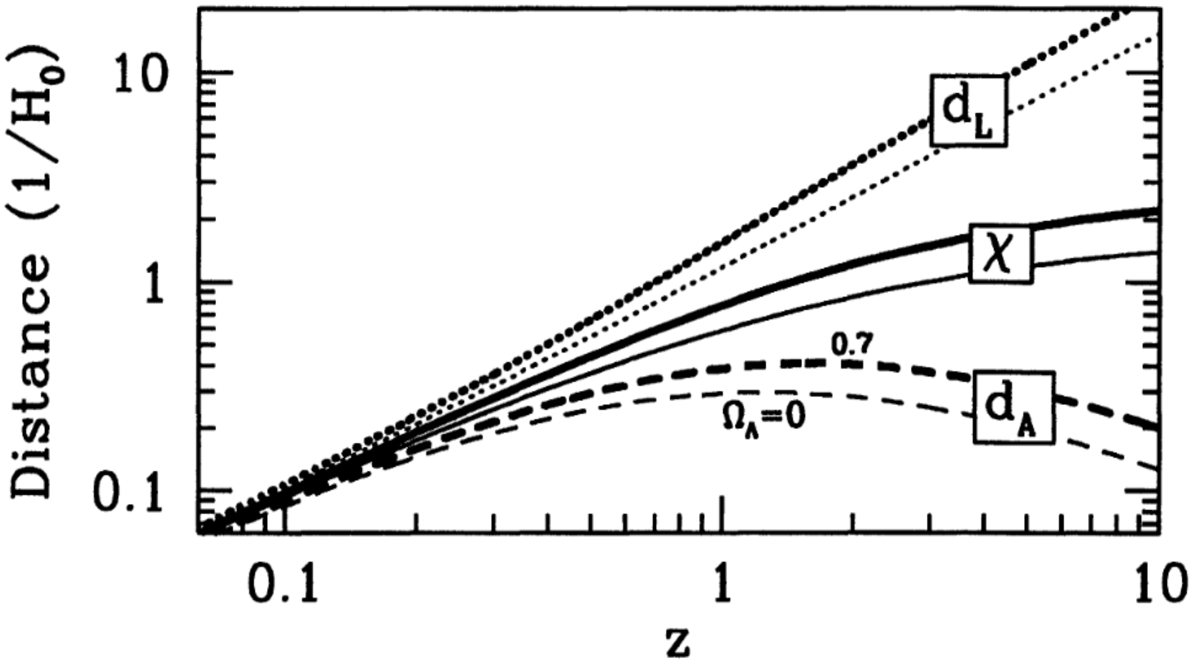



**Figure 2.3.** Three distance measures in a flat expanding universe. From top to bottom, the luminosity distance, the comoving distance, and the angular diameter distance. The pair of lines in each case is for a flat universe with matter only (light curves) and 70% cosmological constant $\Lambda$ (heavy curves). In a $\Lambda$-dominated universe, distances out to fixed redshift are larger than in a matter-dominated universe.

without a cosmological constant. Consequently, objects at a given $z$ appear fainter in a $\Lambda$-dominated Universe compared to what is seen in a matter-dominated Universe, cf. Figure.

## 1.4 Cosmic Inventory (For all we know)

There are a few more details beyond those mentioned above that we should add in view of the subsequent discussions.

First, the matter density $\Omega_M$ decomposes into a baryonic density $\Omega_b$ and a Dark Matter Density $\Omega_d$. Strong evidence for Dark Matter stems from galaxy rotation curves, but CMB & LSS lead to the most precise determinations, $\Omega_b = 0{,}0487$ and $\Omega_d = 0{,}266$, that are known with a relative accuracy at the

percent level. We will see that these two quantities behave in a very different manner at temperatures above the recombination of hydrogen, that allows also to measure $\Omega_b$ and $\Omega_d$ to high accuracy. While the main contribution to $\Omega_b$ is indeed due to protons and neutrons, for an astronomer, also an electron passes as a baryon. We note in passing that considering Big Bang Nucleosynthesis (BBN) is another important (less accurate, but complementary) method of determining $\Omega_b$.
Another apparent fluid component of the unperturbed Universe are the CMB photons at $T = (2{,}7255 \pm 0{,}0006)\,K$ (about $2{,}35 * 10^{-4}\,eV$). Besides, there should be a cosmic neutrino background that has not been observed (because it is out of reach of the sensitivity of even futuristic experiments).
To discuss neutrinos, it is useful to introduce the entropy density

$$s = \frac{S}{V} = \frac{e + P}{T}$$

Now, for a single degree of freedom,

$$s = \frac{1}{T} \int_0^\infty \frac{p^2 dp}{2\pi^2} \frac{4}{3} p \frac{1}{e^{\frac{p}{T}} \mp 1} = \frac{2\pi^2}{45} T^3 * \begin{cases} 1 & \text{for bosons} \\ \frac{7}{8} & \text{for fermions} \end{cases}$$

If the expansion of the Universe proceeds in a thermodynamically reversible manner, the entropy $S$ is conserved, and consequently $s \propto a^{-3}$.
Similarly, we infer that per relativistic degree of freedom

$$e = \int_0^\infty \frac{p^2 dp}{2\pi^2}\, p \, \frac{1}{e^{\frac{p}{T}} \mp 1} = \frac{\pi^2}{30} T^4 * \begin{cases} 1 & \text{for bosons} \\ \frac{7}{8} & \text{for fermions} \end{cases}$$

Now, when the temperature of the Universe falls below 1 MeV reactions maintaining thermal equilibrium of the neutrinos, such as $e^+e^- \leftrightarrow \nu\bar{\nu}$ and $e^-\nu \leftrightarrow e^-\nu$ become slower than the Hubble rate and the neutrinos <u>decouple</u>. Shortly after, the temperature drops below the electron mass and electrons and positrons annihilate, injecting their energy into photons, but not into the neutrinos, that are decoupled. Before the annihilations, the entropy density within photons, electrons and positrons is (at a temperature $T_1$, say)

$$s_1 = \frac{2\pi^2}{45} T_1^3 \left[ 2 + 4 * \frac{7}{8} \right] = \frac{2\pi^2}{45} T_1^3 \, \frac{11}{2}$$

whereas after annihilations (at a temperature $T_\gamma$)

$$s_\gamma = \frac{2\pi^2}{45} T_\gamma^3 \; 2$$

Equating these entropies in the electron-photon sector leads to $s_1 a_1^3 = s_\gamma a_\gamma^3 \implies \dfrac{a_\gamma^3}{a_1^3} = \dfrac{11}{4} \dfrac{T_1^3}{T_\gamma^3}$

At the same time, the neutrino temperature scales down as $\dfrac{T_1}{T_\nu} = \dfrac{a_\gamma}{a_1} = \left(\dfrac{11}{4}\right)^{\frac{1}{3}} \dfrac{T_1}{T_\gamma}$,

what implies: $\dfrac{T_\nu}{T_\gamma} = \left(\dfrac{4}{11}\right)^{\frac{1}{3}}$

Sometimes, the radiation density of the Universe is expressed as

$$\rho_{rad} = \rho_{\gamma}\left[1 + N_{eff}\,\frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}\right]$$

Under the idealised assumptions made above, $N_{eff} = 3$. A more detailed calculation reveals $N_{eff} = 3{,}046$, which may change if there are extra relativistic degrees of freedom or if there is a non-standard heating history. At late times, when the neutrinos become non-relativistic,

$$\Omega_{\nu} = \frac{m_{\nu}}{94 h^2 eV}$$

where $h = \dfrac{H_0}{100\,\frac{km}{s\,Mpc}} \approx 0{,}68$.

As present constraints imply $0{,}05\,eV \lesssim m_{\nu} \ll 1\,eV$, neutrinos do not qualify as non-relativistic Dark Matter.