GENERAL RELATIVITY WS 2017/2018

Technische Universität München

December 17, 2017

Exercise Sheet 9*

The solutions to the following problem set should be handed in by the 8th of January at 8:30 a.m. at the postbox next to PH 3218.

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1. Let (r_0, θ_0, ϕ_0) be the coordinates of a static observer on a Schwarzschild black-hole background of mass M. The observer now drops a lantern directly into the black-hole (radially inwards trajectory). The lantern is set to emit light at fixed wavelength, λ_{em} in its rest frame.

   (a) What is the coordinate speed of the lantern dr/dt?

   (b) What is the proper speed of the lantern as a function of r, i.e. the speed measured by a comoving observer at a given r. Additionally, what is the speed at $r = 2GM$?

   (c) Compute the wavelength λ_{obs} measured by the observer at r_0 as a function of r_{em}, the radius at which the flash was emitted.

   (d) Calculate the time t_{obs} at which a flash emitted at r_{em} will be observed at r_0.

   (e) Show that at late times, the redshift increases exponentially:

   $$\frac{\lambda_{obs}}{\lambda_{em}} \propto e^{\frac{t_{obs}}{T}}. \tag{1}$$

   Find how T depends on M.

2. Show that the capture cross section for a Schwarzschild metric is:

   $$\sigma_{cap} = \pi b_{crit}^2 = \frac{\pi L_{crit}^2}{E^2 - m^2}. \tag{2}$$

   with b the impact parameter, i.e. the radius at which the particle starts increasing its radius after having closing in the body, and "critical" meaning the smallest such b for which the particle would be captured directly (no orbit) when coming from infinity. Then, prove that for the high-energy case one gets:

   $$\sigma_{cap} = 27\pi M^2 \left(1 + \frac{2}{3\tilde{E}^2} + \cdots\right), \tag{3}$$

   where $\tilde{E} = E/m$, and that for the low energy case we have:

   $$\sigma_{cap} = 16\pi \frac{M^2}{\beta^2}, \tag{4}$$

   with $\beta = v/c$.

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3. Derive the formula for the deflection of light on a Schwarzschild background,

$$\Delta\phi = \frac{4GM}{bc^2} \tag{5}$$

by considering light traveling in a geodesic and then approximating to first order in M to deal with the integral giving the deflection (Hint: deal with the integral in terms of the distance of closest approach instead of the impact parameter b, which are slightly different when $M \neq 0$).