GENERAL RELATIVITY WS 2017/2018

Technische Universität München

November 27, 2017

Exercise Sheet 6*

The solutions to the following problem set should be handed in by the 1st of December at 8:30 a.m. at the postbox next to PH 3218.

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1. **[3 Pts.]** Consider the Lagrangian corresponding to electromagnetism in a curved space-time:

$$\mathcal{L}_{EM} = \sqrt{-g}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right) \tag{1}$$

   (a) Compute the energy-momentum tensor.

   (b) Add the following term to the Lagrangian:

$$\mathcal{L}' = \beta R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \tag{2}$$

   How are the equations of motion for the fields in the theory changed?

   (c) Is the current conserved after adding $\mathcal{L}'$?

2. **[4 Pts.]** Consider again the weak field approximated metric:

$$ds^2 = -(1+\Phi)dt^2 + (1-\Phi)(dx^2 + dy^2 + dz^2), \tag{3}$$

with $|\Phi| \ll 1$ being the gravitational potential. Include in the theory an energy-momentum tensor for a perfect fluid:

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + p g^{\alpha\beta}, \tag{4}$$

and assume it moves slowly through space $dx^i/dt \ll 1$ and $p \ll \rho$. Using the fact that the energy-momentum tensor is covariantly conserved, derive the classical mass conservation law and the equation of motion of a fluid subject to a gravitational field, i.e.:

$$\frac{\mathrm{d}}{\mathrm{dt}}\rho = -\rho\frac{\partial v^j}{\partial x^j} \qquad \textit{and} \qquad \rho\frac{\mathrm{d}}{\mathrm{dt}}v^j = -\rho\frac{\partial \Phi}{\partial x^j} - \frac{\partial p}{\partial x^j}, \tag{5}$$

Compute the pressure gradient for the case of Earth and write it in terms of pressure and temperature, for this purpose use the ideal-gas law and the density to be $\rho = \bar{n}\bar{\mu}$, with $\bar{n}$ the number density of molecules and $\bar{\mu}$, as well as $\phi = -\frac{2GM}{r}$. Use the sea level pressure to compute the pressure at the top of mount Everest. (Make a reasonable assumption for the temperature's dependence on altitude)

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3. **[3 Pts.]** Let $\gamma_s(t)$ be a continuous family of geodesics, meaning that for each s, $\gamma_s(t)$ satisfies the geodesic equation. Consider the tangent vectors:

$$T^\mu = \frac{\partial x^\mu}{\partial t} \qquad \text{and} \qquad S^\mu = \frac{\partial x^\mu}{\partial s} \tag{6}$$

Compute the relative acceleration, $A = \frac{D^2}{dt^2} S = (\nabla_T(\nabla_T S))$ and express it in terms of the curvature and T and S.