GENERAL RELATIVITY WS 2017/2018

Technische Universität München

November 8, 2017

Exercise Sheet 4*

The solutions to the following problem set should be handed in by the 17th of November at 8:30 a.m. at the postbox next to PH 3218.

---

1. **(1 pt.)** In $\mathbb{R}^3$, consider a ray $[0,\infty)$ (infinite movable stick) attached perpendicularly to the z axis at 0. Imagine we start rotating the ray around z while we move its endpoint upwards according to a given relation and let $\theta \in [0, 4\pi]$ be the angle by which the ray is rotated. Answer the following questions using geometrical arguments:

   (a) If the relation is $z = \sin^2(\theta/2)$, will the set of points described by the ray be a manifold? If not, can you restrict the set in a clever way such that it is?

   (b) If the relation is $z = \sin(\theta/2)$, will the set of points described by the ray be a manifold? If not, can you restrict the set in a clever way such that it is?

2. **(2 pts.)**Verify explicitly that the bracket of fields $[\cdot,\cdot]$ defined in the lecture fulfills the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{1}$$

   for X, Y, Z being smooth vector fields.

3. **(3 pts.)** Find the expressions for the gradient, $\sum_i \partial_i \Phi dx^i$, the divergence $\partial_i X^i$, and the rotational, with i-th component $\partial_j X_i \, dx^i \wedge dx^j$ in spherical coordinates.

4. **(2 pts.)**Starting from the Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{2}$$

   make the following coordinate change, "rotating coordinate frame",

$$t' = t \tag{3}$$

$$x' = \sqrt{x^2+y^2}\cos(\phi - \omega t) \tag{4}$$

$$y' = \sqrt{x^2+y^2}\sin(\phi - \omega t) \tag{5}$$

$$z' = z \tag{6}$$

   where $\tan\phi = y/x$. Find the components of the metric in these coordinates.

---

*Responsible for the sheet: Juan S. Cruz, Office 1112, juan.cruz@tum.de

5. **(2 pts.)** Prove the following relations between the covariant derivative and the usual derivative:

$$\Gamma^{\alpha}{}_{\beta\alpha} = \partial_\beta(\ln\sqrt{-g}) \tag{7}$$

$$\nabla_\alpha A^\alpha = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}A^\alpha) \tag{8}$$

Notice it also works for antisymmetric (2,0) tensors

$$\nabla_\beta F^{\alpha\beta} = \frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}F^{\alpha\beta}) \tag{9}$$