GENERAL RELATIVITY WS 2017/2018

Technische Universität München

October 23, 2017

Exercise Sheet 2

The solutions to the following problem set should be handed in by 30th of October at 8:30 a.m. at the postbox next to PH 3218.

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1. **(2 pts.)** Prove that the Levi-Civita **symbol** transforms as a tensor density, i.e.

$$\epsilon_{\mu_1'\mu_2'\cdots\mu_n'} = \left|\frac{\partial x^{\mu_i'}}{\partial x^{\mu_j}}\right| \epsilon_{\mu_1\mu_2\cdots\mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \frac{\partial x^{\mu_2}}{\partial x^{\mu_2'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \tag{1}$$

   For the above reason the proper Levi-Civita **tensor** is defined as

$$\varepsilon_{\mu_1\mu_2\cdots\mu_n} = \sqrt{|g|}\epsilon_{\mu_1\mu_2\cdots\mu_n} \tag{2}$$

   where g is the determinant of the metric $(0, 2)$-tensor.

2. **(3 pts.)** Assume a given Lagrangian $\mathcal{L} = \mathcal{L}(\Phi, \partial_\mu\Phi)$ is invariant under translations of the coordinates ($x^\mu \to x^\mu - \epsilon^\mu$) of solutions to the Euler-Lagrange equations. By requiring invariance such invariance on $\mathcal{L}$, show that the following tensor,

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\partial_\nu \Phi - \delta^\mu{}_\nu \mathcal{L}, \tag{3}$$

   referred to as an energy-momentum tensor, is conserved; meaning the following equation holds $\partial_\mu T^\mu{}_\nu = 0$.

   Use the previous results for the case of a Lagrangian invariant under rotation of the coordinates, ($x^\mu \to x^\mu + w^\mu{}_\nu x^\nu$) where $w^\mu{}_\nu$ is an arbitrary antisymmetric matrix.[1]

3. **(2 pts.)** From a finite collection of particles to a fluid. Consider an energy-momentum tensor of the form:

$$T_{\mu\nu} = \sum_a \frac{p_\mu^{(a)} p_\nu^{(a)}}{p^{0(a)}} \delta(\vec{x} - \vec{x}^{(a)}) \tag{4}$$

   Show that by smoothing over world-lines this tensor reduces to the perfect fluid one an **isotropic** velocity distribution.

4. **(3 pts.)** Study the electromagnetic theory Lagrangian plus the term $\mathcal{L}' = \epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$. Find its expression in terms of the classical fields $\vec{E}$ and $\vec{B}$. How are the Maxwell equations modified by this? Show that $\mathcal{L}'$ is a pseudoscalar, meaning that it changes its sign under a parity transformation $x \to -x$. (+1 bonus pt.) Express $\mathcal{L}'$ as a total derivative.

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[1] This corresponds to an infinitesimal rotation, in other words they are elements of the Lie algebra of the group of rotations.