GENERAL RELATIVITY WS 2017/2018
Technische Universität München
January 18, 2018

Exercise Sheet 12*

The solutions to the following problem set should be handed in by Monday the 29th of January at 8:30 a.m. at the postbox next to PH 3218.

Consider a metric $g_{\mu\nu}$ for our space-time and assume it can be split in the following way:

$$g_{\mu\nu} = g^{(B)}_{\mu\nu} + h_{\mu\nu} \tag{1}$$

Assume that the components of $h_{\mu\nu}$ are much smaller than the components of $g^{(B)}_{\mu\nu}$ so that it makes sense to expand the Christoffel symbols and the curvature tensor in powers of $h_{\mu\nu}$. We already now that the difference between covariant derivatives is a tensor (from the problems on the Lagrangian formulation and Palatini method)

$$S^{\mu}{}_{\beta\gamma} = \Gamma^{\mu}{}_{\beta\gamma} - \Gamma^{(B)\mu}{}_{\beta\gamma} \tag{2}$$

1. Prove the following equation

$$g^{\mu\nu} = g^{(B)\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h_{\alpha}{}^{\nu} - h^{\mu\alpha}h_{\alpha}{}^{\beta}h_{\beta\gamma}g^{\gamma\nu} \tag{3}$$

   from which one can deduce:

$$g^{\mu\nu} = g^{(B)\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h_{\alpha}{}^{\nu} - h^{\mu\alpha}h_{\alpha}{}^{\beta}h_{\beta}{}^{\nu} + \cdots \tag{4}$$

2. Use a local Lorentz frame for the background metric to compute $S^{\mu}{}_{\beta\gamma}$, and then transform back to the original frame to obtain

$$S^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\mu\alpha}\left(h_{\alpha\beta|\gamma} + h_{\alpha\gamma|\beta} - h_{\beta\gamma|\alpha}\right), \tag{5}$$

   where $_|$ means covariant derivative with respect to the background metric, not the full metric.

3. Employ the same method to compute the difference of the full Riemann tensor and the background Riemann tensor in terms of $S^{\alpha}{}_{\beta\gamma}$. Find the difference of the Ricci tensors as well.

4. Use the equation from (5) to write down the difference of the Ricci tensors in terms of $h_{\mu\nu}$:

$$R_{\alpha\beta} - R^{(B)}_{\alpha\beta} = R^{(1)}_{\alpha\beta}(h_{\mu\nu}) + R^{(2)}_{\alpha\beta}(h_{\mu\nu}), \tag{6}$$

   here $R^{(1)}_{\alpha\beta}(h_{\mu\nu})$ contains only linear terms in $h_{\mu\nu}$ and $R^{(2)}_{\alpha\beta}(h_{\mu\nu})$ contains only quadratic terms of $h_{\mu\nu}$.

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*Responsible for the sheet: Juan S. Cruz, Office 1112, juan.cruz@tum.de