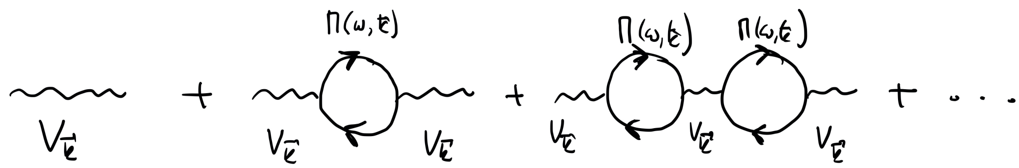
Problem set 4: Screening of the Coulomb potential in metals

Due date: Dec 11

Electrons in a metal screen electrostatic potential created by external impurities. In this problem set you are asked to determine the static limit $\omega = 0$ of the screened Coulomb potential in the low-momentum regime $k \ll p_F$, where p_F is the Fermi momentum. One can obtain the screened potential by summing a series of Feynman diagrams



Here the wiggly line denotes $V_{\mathbf{k}} = 4\pi e^2/k^2$ which is the Fourier transform of the bare Coulomb potential and the fermion loop $\Pi(\omega, \mathbf{k})$ is the polarization operator that you will be asked to compute.

1. From the figure, the screened Coulomb potential is

$$\tilde{V}_{\omega,\mathbf{k}} = V_{\mathbf{k}} + V_{\mathbf{k}}\Pi(\omega,\mathbf{k})V_{\mathbf{k}} + V_{\mathbf{k}}\Pi(\omega,\mathbf{k})V_{\mathbf{k}}\Pi(\omega,\mathbf{k})V_{\mathbf{k}} + \ldots \tag{1}$$

   Sum this series.

2. To determined the static limit $\omega = 0$ of the screened potential we must calculate the static polarization operator $\Pi(\omega = 0, \mathbf{k})$. It is given by the loop integral

$$\Pi(\omega = 0, \mathbf{k}) = -2i \int G(\epsilon, \mathbf{p}_-)G(\epsilon, \mathbf{p}_+)\frac{d\epsilon d^3p}{(2\pi)^4}, \tag{2}$$

   where $\mathbf{p}_\pm = \mathbf{p} \pm \mathbf{k}/2$ and we introduced the fermionic Green's function in momentum space

$$G(\epsilon, \mathbf{p}) = \frac{1}{\epsilon - \xi_{\mathbf{p}} + i\delta \text{sign}(\xi_{\mathbf{p}})}. \tag{3}$$

   Here $\xi_{\mathbf{p}} = (p^2 - p_F^2)/(2m)$ is the Fermi energy measured from the Fermi level $\xi_F = p_F^2/(2m)$ and δ is an infinitesimal positive constant. Using the Cauchy residue theorem perform first the frequency integral and show that

$$\Pi(\omega = 0, \mathbf{k}) = 2\int \frac{n(\mathbf{p}_-) - n(\mathbf{p}_+)}{\xi(\mathbf{p}_-) - \xi(\mathbf{p}_+)}\frac{d^3p}{(2\pi)^3}, \tag{4}$$

   where we introduced the Fermi-Dirac distribution $n(\mathbf{p}) = \theta(p_F - p)$.

3. Show that in the regime $k \ll p_F$ one has approximately

$$n(\mathbf{p}_-) - n(\mathbf{p}_+) = k\cos\theta\delta(p - p_F), \tag{5}$$

   where θ is the angle between the vectors $\mathbf{p}$ and $\mathbf{k}$. Using this in Eq. (4) demonstrate that approximately

$$\Pi(\omega = 0, \mathbf{k}) = -\nu, \tag{6}$$

   where $\nu = mp_F/\pi^2$ is the total density of states at the Fermi level.

4. Using now the result (6) show that the screened static Coulomb potential is

$$\tilde{V}_{\omega=0,\mathbf{k}} = \frac{4\pi e^2}{k^2 + \kappa^2}, \tag{7}$$

where $\kappa^2 = 4\pi e^2 \nu$. Fourier transform this expression to real space, how does the screened potential decay?

5. Screening also takes place in a classical plasma. Consider a static charge $\rho_0(\mathbf{r})$ embedded into the plasma. The resulting electrostatic potential $\Phi(\mathbf{r})$ satisfies the Poisson equation

$$\nabla^2 \Phi(\mathbf{r}) = -4\pi(\rho_0(\mathbf{r}) + \delta\rho(\mathbf{r})), \tag{8}$$

where the density of the screening charge $\delta\rho(\mathbf{r})$ is fixed by the electrostatic potential $\Phi(\mathbf{r})$ via the Boltzmann equation

$$\delta\rho(\mathbf{r}) = en(e^{-e\Phi(\mathbf{r})/T} - 1). \tag{9}$$

Linearize the resulting Poisson equation and determine the screened potential of a point charge $\rho_0(\mathbf{r}) \sim \delta(\mathbf{r})$. Compare the result with the one obtained above for a zero-temperature metal.