Problem set 5: Energy spectrum of spinless Luttinger liquid

Due date: Jan 15

Landau Fermi liquid theory is a powerful paradigm to study normal interacting Fermi systems. In one spatial dimension, however, fermionic quasiparticle excitations decay too fast into particle-hole pairs and a bosonic Luttinger liquid emerges. In this problem set you are asked to diagonalize the Hamiltonian of the spinless Luttinger liquid and determine the dispersion relation of the collective particle-hole excitation.
Consider an interacting system of spinless fermions governed by the Hamiltonian

$$\widehat{\mathcal{H}} = \frac{1}{L}\sum_p \xi(p) a_p^+ a_p + \frac{1}{2L^2}\sum_{p_1+p_3=p_2+p_4} V_{p_1-p_2} a_{p_1}^+ a_{p_2} a_{p_3}^+ a_{p_4}, \tag{1}$$

where $\xi(p) = (p^2 - p_0^2)/(2m)$ and V_k is the Fourier transform of the interaction potential, whose range r_0 is assumed to be much larger than the inverse of the Fermi momentum p_0.
We are interested only in states which are close to the Fermi level. In the lecture we demonstrated that the free part of the Hamiltonian can be written as

$$\widehat{\mathcal{H}}_0 = \pi v \sum_k \left[\widehat{\rho}_R(k)\widehat{\rho}_R(-k) + \widehat{\rho}_L(k)\widehat{\rho}_L(-k)\right], \tag{2}$$

where the Fermi velocity $v = p_0/m$ and we introduced the right- and left-moving density operators

$$\widehat{\rho}_R(k) = \frac{1}{L}\sum_{p>0} a_{p-k/2}^+ a_{p+k/2}, \quad \widehat{\rho}_L(k) = \frac{1}{L}\sum_{p<0} a_{p-k/2}^+ a_{p+k/2}. \tag{3}$$

As for the interaction term, it is enough to consider only scattering with momenta close to the Fermi points, examples are shown in the figure

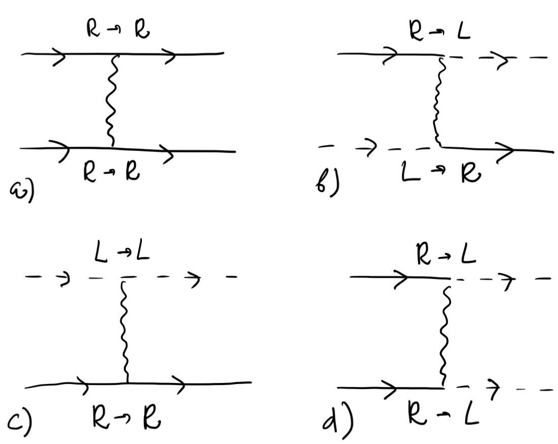

Figure 1: Different types of scattering processes close to the Fermi points, right- and left-moving particles are displayed with solid and dashed lines, respectively.

1. Which of the processes give rise to density-density interactions, i.e. interactions that can be written in terms of the densities (3) only?

2. The process d) called the Umklapp process does not conserve total momentum during the collision. Can you, however, imagine any system, where this process can happen?

3. Consider a model, where the interaction term reduces to the following pure density-density form

$$\widehat{\mathcal{H}}_{int} = \frac{1}{2}\sum_k g_1(k)(\widehat{\rho_R}(k)\widehat{\rho_R}(-k) + \widehat{\rho_L}(k)\widehat{\rho_L}(-k)) + \sum_k g_2(k)\widehat{\rho}_R(k)\widehat{\rho}_L(-k). \tag{4}$$

Using the decomposition of the density operators into the bosonic creation and annhilation operators

$$\begin{aligned} \widehat{\rho}_R(x) &= \textstyle\sum_{k>0} \frac{1}{\lambda_k L}\left(b_k e^{ikx} + b_k^+ e^{-ikx}\right) \\ \widehat{\rho}_L(x) &= \textstyle\sum_{k<0} \frac{1}{\lambda_k L}\left(b_k e^{ikx} + b_k^+ e^{-ikx}\right) \end{aligned} \tag{5}$$

with $\lambda_k = (2\pi/|k|L)^{1/2}$, express the total Hamiltonian $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_0 + \widehat{\mathcal{H}}_{int}$ as

$$\widehat{\mathcal{H}} = \frac{1}{2\pi L}\sum_{k>0}\left[(2\pi k v + k g_1(k))\left(b_k^+ b_k + b_{-k}^+ b_{-k}\right) + k g_2(k)\left(b_k^+ b_{-k}^+ + b_k b_{-k}\right)\right]. \tag{6}$$

4. Diagonalize the Hamiltonian (6) using the Bogoliubov transformation

$$\begin{aligned} \widetilde{b}_k &= \operatorname{ch}\theta_k b_k + \operatorname{sh}\theta_k b_{-k}^+, \\ \widetilde{b}_{-k}^+ &= \operatorname{ch}\theta_k b_{-k}^+ + \operatorname{sh}\theta_k b_k \end{aligned}. \tag{7}$$

Compute the commutators of the Bogoliubov creation and annihilation operators. Extract the dispersion relation of excitations $\omega(k)$ from the diagonal form of the Hamiltonian

$$\widehat{\mathcal{H}} = \frac{1}{L}\sum_k \omega(k)\widetilde{b}_k^+\widetilde{b}_k. \tag{8}$$

Discuss your result for the dispersion relation.