# Problem set 3: Collective modes of a metal from the conductivity tensor

*Due date: Nov 27*

In a translation-invariant system, the conductivity tensor $\sigma^{ij}(\omega, \mathbf{q})$ is the response function which quantifies linear response of the electric current $j^i(\omega, \mathbf{q})$ to the external electric field $E_j(\omega, \mathbf{q})$ via the formula $j^i(\omega, \mathbf{q}) = \sigma^{ij}(\omega, \mathbf{q}) E_j(\omega, \mathbf{q})$. In an isotropic medium that is invariant under time-reversal transformation the conductivity tensor can be decomposed into the longitudinal and transverse parts

$$\sigma^{ij}(\omega, \mathbf{q}) = \hat{q}^i \hat{q}^j \sigma_l(\omega, \mathbf{q}) + (\delta^{ij} - \hat{q}^i \hat{q}^j) \sigma_t(\omega, \mathbf{q}), \tag{1}$$

where $\hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|$.

1. Demonstrate that the tensors $\hat{q}^i \hat{q}^j$ and $(\delta^{ij} - \hat{q}^i \hat{q}^j)$ are two complementary projector operators.

2. Assuming the hydrodynamic equation of motion for the electric current density $\mathbf{j}$ and the particle density $n$

$$\partial_t \mathbf{j}(t, \mathbf{x}) = -\frac{e}{m} \frac{\partial p}{\partial n} \nabla n(t, \mathbf{x}) - \frac{1}{\tau} \mathbf{j}(t, \mathbf{x}) + \frac{ne^2}{m} \mathbf{E}(t, \mathbf{x}) \tag{2}$$

   compute $\sigma_l(\omega, \mathbf{q})$ and $\sigma_t(\omega, \mathbf{q})$. Along the way, use the continuity equation which in Fourier space reads[1] $e\omega n(\omega, \mathbf{q}) = \mathbf{q} \cdot \mathbf{j}(\omega, \mathbf{q})$ and remember that $\partial p / \partial n = m c_s^2$ with $c_s$ being the velocity of sound. Extract the dispersion relation of the longitudinal collective mode $\omega_l(\mathbf{q})$ from setting the denominator of the longitudinal conductivity to zero. Analyze its behavior for $\tau = 0$ and $\tau \neq 0$. What about the dispersion relation of the transverse collective mode?

3. The dispersion relation of the collective modes change qualitatively if the Coulomb interaction between fermions is introduced. Take here as given that in this case the dispersions of the longitudinal and transverse modes can be obtained from solving the equations $i\omega - 4\pi\sigma_l(\omega, \mathbf{q}) = 0$ and $\omega(i\omega - 4\pi\sigma_t(\omega, \mathbf{q})) = ic^2\mathbf{q}^2$, where $\sigma_l$ and $\sigma_l$ are the conductivities computed in the previous point and $c$ is the velocity of light which is much larger than the speed of sound $c_s$. Determine the dispersion of the longitudinal mode and analyze its asymptotic behavior in the regimes of small and large momenta $\mathbf{q}$, assuming the square of the plasma frequency $\omega_p^2 = 4\pi n e^2/m > 1/(4\tau^2)$. Determine the dispersion of the transverse mode, assuming that $\tau \to \infty$, and analyze its asymptotics.

4. Imagine that an external constant magnetic field which brakes time-reversal symmetry is switched on. Is the decomposition of the conductivity tensor (1) still valid? If not, what kind of terms are now allowed?

[1] Our Fourier conventions are $f(t, \mathbf{x}) = \int \frac{d\omega}{2\pi} \frac{d^3q}{(2\pi)^3} f(\omega, \mathbf{q}) e^{-i(\omega t - \mathbf{q}\cdot\mathbf{x})}$.