Problem set 6: Interacting Bose gas

Due date: Jan 31

Consider a zero-temperature three-dimensional Bose gas with short-range interactions described by the Euclidean action

$$S\left[\psi^{*},\psi\right]=\int d\tau\int d^{3}r\left\{\psi^{*}(\mathbf{r},\tau)\left(\partial_{\tau}-\mu-\frac{\nabla^{2}}{2m}\right)\psi(\mathbf{r},\tau)+\frac{g}{2}\psi^{*}(\mathbf{r},\tau)\psi^{*}(\mathbf{r},\tau)\psi(\mathbf{r},\tau)\psi(\mathbf{r},\tau)\right\} \tag{1}$$

where $\psi(\mathbf{r},\tau)$ is a bosonic field.

1. What is the expression of the action in a saddle-point approximation, where the field $\psi(\mathbf{r},\tau)=\psi_0$ is assumed to be uniform and time-independent. Under which condition do we have $\psi_0\neq 0$? What is the symmetry which is broken spontaneously? Why can we choose ψ_0 real without loss of generality?

2. We now consider fluctuations $\psi'(\mathbf{r},\tau)=\psi(\mathbf{r},\tau)-\psi_0$ of the bosonic field around its saddle-point value ψ_0. What is the action $S\left[\psi^{*},\psi\right]$ to quadratic order in the fluctuating field ψ'?

3. Introduce the two-component Bogoliubov field

   $$\Psi\left(\mathbf{k},i\omega\right)=\begin{pmatrix}\psi'\left(\mathbf{k},i\omega\right)\\ \psi'^{*}\left(-\mathbf{k},-i\omega\right)\end{pmatrix},\qquad \Psi^{\dagger}\left(\mathbf{k},i\omega\right)=\left(\psi'^{*}\left(\mathbf{k},i\omega\right),\psi'\left(-\mathbf{k},-i\omega\right)\right), \tag{2}$$

   where

   $$\begin{aligned}\psi'\left(\mathbf{k},i\omega\right)&=\int d\tau\int d^{3}r e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega\tau)}\psi'(\mathbf{r},\tau),\\ \psi'^{*}\left(\mathbf{k},i\omega\right)&=\int d\tau\int d^{3}r e^{i(\mathbf{k}\cdot\mathbf{r}-\omega\tau)}\psi'^{*}(\mathbf{r},\tau).\end{aligned} \tag{3}$$

   Show that we can write the quadratic action in the form

   $$S_{quad}=S_0+\frac{1}{2}\int\frac{d\omega}{2\pi}\frac{d^{3}k}{(2\pi)^{3}}\Psi^{\dagger}\left(\mathbf{k},i\omega\right)\mathcal{D}\left(\mathbf{k},i\omega\right)\Psi\left(\mathbf{k},i\omega\right), \tag{4}$$

   where S_0 is the saddle-point value and $\mathcal{D}\left(\mathbf{k},i\omega\right)$ is a 2×2 matrix. What is the excitation spectrum in the superfluid phase, where $\psi_0\neq 0$? Does it agree with Goldstone theorem?

4. Using the quadratic action S_{quad}, compute the partition function $Z=\int\mathcal{D}\psi'\mathcal{D}\psi'^{*}e^{-S_{quad}}\equiv e^{-S_{eff}}$. Show that the effective action is

   $$S_{eff}=S_0+\frac{1}{2}\Omega\int\frac{d\omega}{2\pi}\frac{d^{3}k}{(2\pi)^{3}}\ln\det\mathcal{D}\left(\mathbf{k},i\omega\right), \tag{5}$$

   where Ω is the spacetime volume. Perform the frequency integration for the density

   $$n=-\frac{1}{\Omega}\frac{\partial S_{eff}}{\partial\mu}. \tag{6}$$