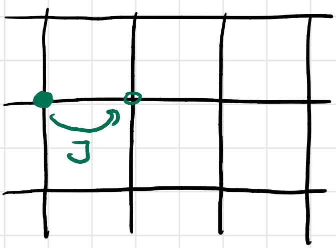
Boson-vortex duality

We emphasized that superconductors and superfluids are very different at low energy (gapped vs gapless), but there is an intriguing relation (duality) between them in two dimensions.

Consider a bosonic Hubbard model on a square lattice - hopping particles with point-like repulsion



$$\hat{H} = -J \sum_{\langle i,j \rangle} \left(\hat{b}_i^\dagger \hat{b}_j + h.c. \right) + \frac{U}{2} \sum_i \hat{n}_i \left(\hat{n}_i - 1 \right) - \mu \sum \hat{n}_i$$

where $\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i$

1) $J = 0$ lattice sites decouple → single site problem

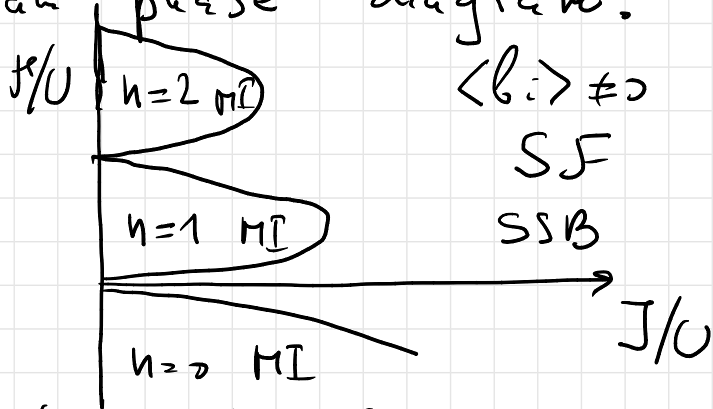
$n = 0$ $\mu < 0$ $\quad$ $n = 1$ $0 < \mu < U$ $\quad$ $n = 2$ $U < \mu < 2U$

integer occupation, filling changes in discrete jumps
↑
Mott insulator

2) $U = 0$ → free boson lattice gas

adding $U \ll J$ ⇒ bosonic superfluid

Quantum phase diagram:



see Sachdev book

We will do the following simplification

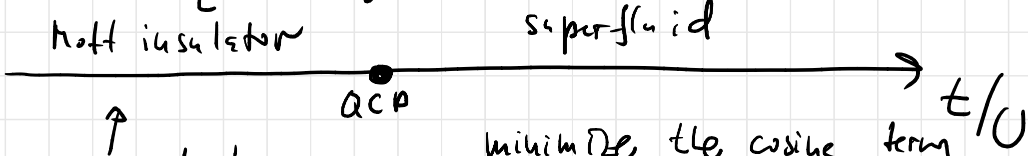
$$\hat{b}_i = e^{i\hat{\varphi}} \sqrt{\hat{n}_i} \qquad [\hat{n}_i, \hat{\varphi}_i] = -i$$

Close to the quantum phase transition between the $U(1)$ preserving MI phase and the $U(1)$ SSB superfluid phase

$$\hat{n} = n_0 + \delta\hat{n} \qquad n_0 = \frac{\tilde{\mu}}{U}$$

approximate Hamiltonian

$$\hat{H} = -\underbrace{2 J n_0}_{t} \sum_{\langle i j \rangle} \cos(\varphi_i - \varphi_j) + \frac{U}{2} \sum (\delta \hat{n})^2$$



minimize int. term

n – integer filling

sharp particle number per site, phase is uncertain

minimize the cosine term

phase is sharp in GS

particle number per site is uncertain

$$\cos(\varphi_i - \varphi_j) \to \frac{1}{2}(\varphi_i - \varphi_j)^2$$

Close to the MI/SF quantum critical point in 2d one can write a continuum field theory of low-energy physics in two very different ways:

1) In terms of original bosons $\vec{b}$: complex scalar

$$\mathcal{L}_{x_4} = |\partial_\mu b|^2 + m^2 |b|^2 + u |b|^4$$

$m^2 < 0$ SSB superfluid phase

$m^2 > 0$ normal Mott insulator phase

as MI → SF, global $U(1)_N$ symmetry undergoes SSB at $T=0$

2) In terms of vortices $\vec{v}$: complex scalar

start in the SSB superfluid phase

vortices are high-energy excitations in this phase, try to "condense" them

Remember however in 2d

*) two pointlike vortices interact

$$V(r) \sim -q_1^v q_2^v \ln r$$

*) experience the Magnus force:

single vortex Lagrangian $\quad L_X = -\pi\, n_s\, \epsilon_{ij} X_i \dot{X}_j$

effective magnetic field

In 2d vortices behave like point-like charged particles that live in magnetic field $B_v \sim n_s$, we cannot SSB condense them, but can <u>Higgs them</u>

$$\mathcal{L}_{AH} = \left|(\partial_\mu - i A_\mu)\vartheta\right|^2 + \bar{m}^2 |\vartheta|^2 + \bar{u}\, |\vartheta|^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Mott insulator $\qquad$ superfluid

GCP $\qquad$ t/U

$\langle \vartheta \rangle \neq 0$

$m^2 < 0$

Higgs phase

$\langle \vartheta \rangle = 0$

$\bar{m}^2 > 0$ $\quad$ vortices are gapped, can be ignored

$\Downarrow$

2+1 dim Maxwell ED with massless photon excitation

$\parallel$

Goldstone of $U(1)_N$ SSB

Dictionary of boson/vortex duality:

XY model	Abelian Higgs model
$j^\mu = b^* \overleftrightarrow{\partial_\mu} b$	$j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$
boson b	instanton of A_μ (destroys flux of A)
2π vortex of b	v
time reversal : $\varphi \to -\varphi$, $b = \sqrt{n}\, e^{i\varphi}$	time reversal $V \to V^*$

Numerous checks of this duality close to the critical point (Monte-Carlo, ...)

Rigorous duality was defined <u>on a lattice</u>

Peskin 1978 Dasgupta/Halperin 1981