Expectation values: $\theta = \hat{\psi}_i^+ \theta_{ij} \psi_j$

$$\langle \hat{\theta} \rangle_{BCS} = \langle BCS | \hat{\psi}_i^+ \theta_{ij} \hat{\psi}_j | BCS \rangle =$$

$$= \langle BCS | (v_{im} b_m + u^*_{im} b_m^+) \theta_{ij} (u_{jn} b_n + v^*_{jn} b_n^+) | BCS \rangle$$

$$= v_{im} \theta_{ij} v^*_{jm}$$

What about Cooper pairing?

Usually all b-operators in

$$|BCS\rangle = \prod_n b_n |0\rangle$$

can be paired up.

Example: s-wave superconductor in continuum

$$|BCS\rangle = \prod_{\vec{k}} b_{\vec{k}\uparrow} b_{-\vec{k}\downarrow} |0\rangle$$

pairing of $\vec{k}\uparrow$ and $-\vec{k}\downarrow$ fermions

when expressed in terms of the original fermions

$$\begin{pmatrix} b_{k\uparrow} \\ b^+_{k\downarrow} \end{pmatrix} = \begin{pmatrix} \cos \theta_k/2 & \sin \theta_k/2 \\ -\sin \theta_k/2 & \cos \theta_k/2 \end{pmatrix} \begin{pmatrix} \psi_{k\uparrow} \\ \psi^+_{-k\downarrow} \end{pmatrix}$$

$$|BCS\rangle \sim \prod_k (1 - \tan \theta_k/2 \, \psi^+_{k\uparrow} \psi^+_{-k\downarrow}) |0\rangle$$

$$= \prod_k \exp(\underbrace{\varphi_k}_{-\tan \theta_k/2} \psi^+_{k\uparrow} \psi^+_{-k\downarrow}) |0\rangle$$

coherent state

Linear superposition of Fock states with 0, 2, 4, ... particles since the mean-field Hamiltonian does not commute with $\hat{N}$.

Two questions: 1) What is the physical meaning of θ_k?

2) How to fix the gap Δ?

We will use Anderson pseudospin representation.

In Nambu space $\Psi_k = \begin{pmatrix} \psi_{k\uparrow} \\ \psi^+_{-k\downarrow} \end{pmatrix}$

$$H_{MF} = \sum_k \Psi_k^+ \vec{h}_k \cdot \vec{\tau} \, \Psi_k + \Omega \frac{\Delta^* \Delta}{g_0}$$

where $\vec{h}_k = (\mathrm{Re}\,\Delta, -\mathrm{Im}\,\Delta, \epsilon_k)$

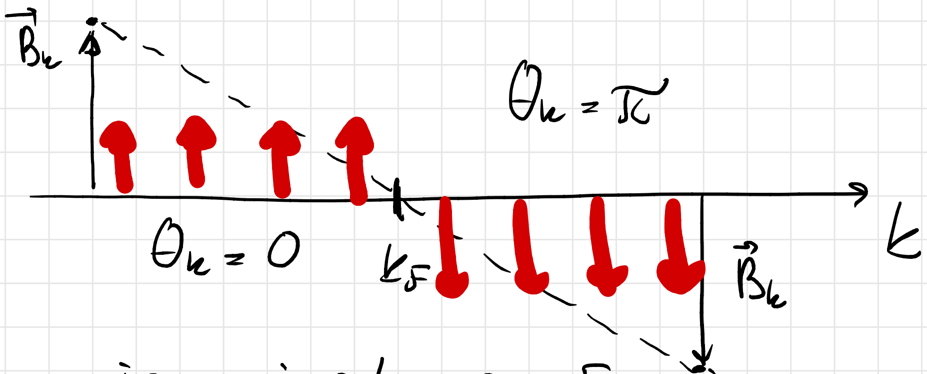
Formally equivalent to the Zeeman term for every k

$$H = -\sum_k \vec{B}_k \vec{S}_k \qquad \vec{B}_k = -\vec{h}_k, \quad \vec{S}_k = \Psi_k^+ \vec{\tau}\, \Psi_k$$

1) Isospins should line up with B_k

2) Excitations are isospin flips

Normal metal: $\vec{B}_k = (0, 0, -\epsilon_k)$



This is just a fermi sea since

$$S_{3k} = \Psi_k^\dagger \tau_3 \Psi_k = (n_{k\uparrow} + n_{-k\downarrow} - 1)$$

and thus

$$|\Uparrow\rangle_k = \psi_{k\uparrow}^\dagger \psi_{-k\downarrow}^\dagger |0\rangle_k$$

$$|\Downarrow\rangle_k = |0\rangle_k$$

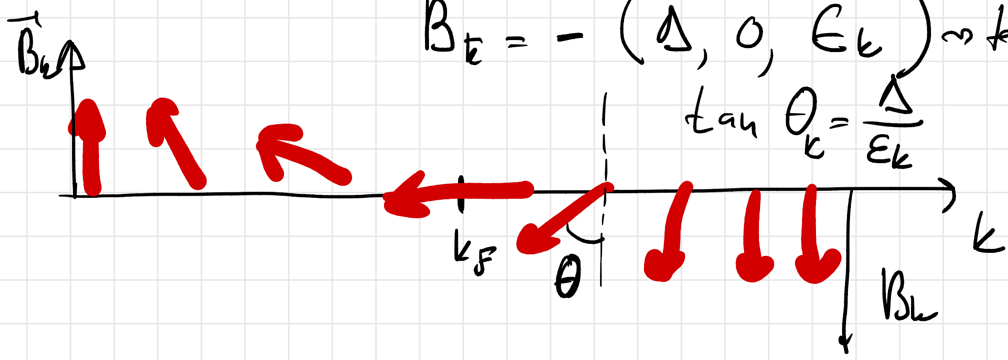
This state is gapless since $B_{k_F} = 0$ and thus there is no penalty for flipping pseudospin at k_F

Paired state

consider $\operatorname{Im} \Delta = 0$

$$B_k = -\underbrace{(\Delta, 0, \epsilon_k)}_{} \rightsquigarrow \text{texture}$$

$$\tan \theta_k = \frac{\Delta}{\epsilon_k}$$



1) Gapped state since $|\vec{B}_k| \neq 0$ for any $\vec{k}$

2) Nontrivial pairing since $\langle \tau_1 \rangle \neq 0$

$$\langle \tau_{1k} \rangle = \langle \Psi_k^+ \tau_1 \Psi_k \rangle$$

$$= \langle (\psi_{k\uparrow}^+, \psi_{-k\downarrow}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_{k\uparrow} \\ \psi_{-k\downarrow}^+ \end{pmatrix} \rangle$$

$$= \langle \psi_{k\uparrow}^+ \psi_{-k\downarrow}^+ + \psi_{-k\downarrow} \psi_{k\uparrow} \rangle =$$

$$= -\sin \theta_k = -\frac{\Delta}{\sqrt{\epsilon_k^2 + \Delta^2}}$$

BCS wave-function: start from $|0\rangle = |\downarrow\rangle$ at every k and rotate by angle θ_k around y-axis

$$|\theta_{\vec{k}}\rangle = \exp\left(-i\,\theta_k \frac{\Psi_k^+ \tau_y \Psi_k}{2}\right) |\downarrow_k\rangle$$

$$= \cos\frac{\theta_k}{2} |\downarrow_k\rangle - \sin\frac{\theta_k}{2} |\uparrow_k\rangle$$

$$|BCS\rangle = \prod_k |\theta_k\rangle \sim \prod_k \left(1 - \tan\frac{\theta_k}{2} \psi_{k\uparrow}^+ \psi_{-k\downarrow}^+\right)|0\rangle$$

which coincides with what was found before what is the physical value of the gap?

Ground state has minimal energy

$$\frac{\partial H_{MF}}{\partial \Delta} = 0 \Rightarrow \Delta = -\frac{g_0}{\Omega}\sum_{\vec{k}} \langle \underbrace{\psi_{-\vec{k}\downarrow}\psi_{k\uparrow}}_{-\frac{1}{2}\sin\theta_k} \rangle$$

$$= +\frac{g}{2\Omega}\sum_{\vec{k}} \frac{\Delta}{\sqrt{\epsilon_k^2+\Delta^2}}$$

BCS gap equation:

$$\boxed{\Delta = g_0 \int \frac{d^3k}{(2\pi)^3} \frac{\Delta}{2\sqrt{\epsilon_k^2+\Delta^2}}}$$

UV divergent integral

in SC $|\epsilon_k| < \omega_D$ → Debaye frequency

$$1 = g_0 N_F \int_{-\omega_D}^{\omega_D} d\epsilon \frac{1}{2\sqrt{\epsilon^2+\Delta^2}} \underset{\omega_D \gg \Delta}{\simeq} g N_F \ln\left(\frac{2\omega_D}{\Delta}\right)$$

$$\Downarrow$$

$$\boxed{\Delta = 2\omega_D e^{-\frac{1}{g_0 N_F}}}$$

the gap is non-perturbative in g_0 and N_F

in cold atoms replace ω_D → by Λ^2,

where $\Lambda \sim 1/r_0$ → range of interaction