# Superfluids and superconductors

Two related phenomena which however differ substantially at low energies

Superconductivity – discovered in the lab of Kamerlingh Onnes in 1911

many-body phenomena of electrically charged particles, such as electrons

* below $T_c$ – strictly zero resistivity

* response to magnetism:

type I – expulsion – Meissner effect

type II – Abrikosov lattice of vortices

quantum mechanics is essential

high $T_c$ SC: largest $T_c \simeq 140\,K$ at the atmosphere

cuprates debates of SC mechanism

Superfluidity – many-body phenomena of electrically neutral particles, e.g. atoms

Experimentally discovered by Kapitsa and Allen in 1937

in fact Kamerlingh Onnes lab cooled $^4He$ below $T_c$, but they did not recognize its significance.

* fluid flow with no resistance
* under rotation - lattice of quantized vortices

1937 - $^4He$, 1972 - $^3He$,

1995 - cold atom BEC, 2000 - fermionic superfluids

Despite similarities between SC and SF, at low energies they are very different

SF - gapless due to spontaneous symmetry breaking (SSB) of global U(1) particle number symmetry

SC - gapped due to the Higgs mechanism of local U(1) electric "symmetry"

Both however exhibit an energy gap in the spectrum of <u>fermionic</u> excitations. This gap is explained by the BCS theory.

## Basics of BCS theory

We start from a simple model of fermions with short-range <u>attractive</u> interactions: $\sigma = \uparrow, \downarrow$ $\quad g > 0$

$$\hat{H} = \int \hat{\psi}_\sigma^+ \left(-\frac{\Delta}{2m} - \mu\right) \hat{\psi}_\sigma - \frac{g}{\Omega} \hat{\psi}_\uparrow^+ \hat{\psi}_\downarrow^+ \hat{\psi}_\downarrow \hat{\psi}_\uparrow$$

Cooper discovered that in the presence of a rigid Fermi sphere, <u>any</u> attractive interaction ($g>0$) gives rise to a <u>two-body bound state</u> (Cooper pair). This is a nontrivial consequence of kinematics of the Fermi surface (finite density of states). In vacuum ($\mu = 0$) one needs a finite $g_{cr} > 0$ in 3d to create a two-body bound state.

Cooper result suggests that the Fermi sea is unstable with respect to formation of Cooper pairs → BCS wave function

The essense of the BCS approach: imagine that in the ground state

$$-\frac{g}{\Omega}\langle \hat{\psi}_\uparrow \hat{\psi}_\downarrow \rangle_{GS} = \underset{\text{SC gap}}{\Delta} \neq 0$$

Cooper pair operator has finite expectation value in the GS. Notice that the GS does not have fixed number of particles.

<u>General analysis</u>:

$\hat{\psi}_i$ – second-quantized fermion operator
destroys a fermion in a quantum state $i$

$i$ is spin, momentum, something else $i = 1, \ldots N$

$\{\hat{\psi}_i, \hat{\psi}_j\} = \{\hat{\psi}_i^+, \hat{\psi}_j^+\} = 0 \qquad \{\psi_i, \psi_j^+\} = \delta_{ij}$

Mean-field approximation: $\hat{\psi}_\uparrow \hat{\psi}_\downarrow = \Delta + (\underbrace{\hat{\psi}_\uparrow \hat{\psi}_\downarrow - \Delta}_{\text{small}})$

$$\hat{H}_{MF} = \hat{\psi}_i^+ \underset{\substack{\text{Hermitian}\\ \text{matrix}}}{H_{0ij}} \hat{\psi}_j + \frac{1}{2}\left(\underset{\substack{\text{antisym.}\\ \text{matrix}}}{\Delta_{ij}} \hat{\psi}_i^+ \hat{\psi}_j^+ + \Delta_{ij}^+ \hat{\psi}_i \hat{\psi}_j\right) + \frac{1}{g_0}\Delta_{ij}^+ \Delta_{ji}$$

$H_{MF}$ does not conserve the particle number.
It is quadratic and can be diagonalized using the Nambu trick

$$H_{MF} = \frac{1}{2}\left(\psi_i^\dagger, \psi_i\right) \underbrace{\begin{pmatrix} H^0_{ij} & \Delta_{ij} \\ \Delta^\dagger_{ij} & -H^{0T}_{ij} \end{pmatrix}}_{\mathcal{H}_{BdG} \to 2N\times 2N \text{ matrix}} \begin{pmatrix} \psi_j \\ \psi_j^\dagger \end{pmatrix} + \frac{1}{2}\operatorname{tr} H^0 + \frac{1}{\Omega}\Delta^\dagger\Delta$$

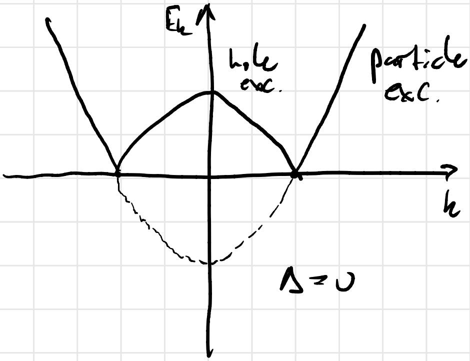
Now solve the single-particle BdG problem

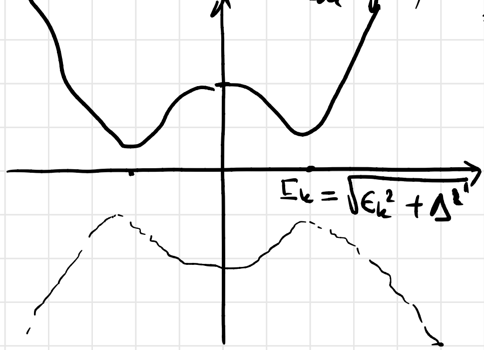
$$\underbrace{\begin{pmatrix} H^0 & \Delta \\ \Delta^\dagger & -H^{0T} \end{pmatrix}}_{2N\times 2N \text{ matrix}} \begin{pmatrix} \vec{u}_m \\ \vec{v}_m \end{pmatrix} = E_m \begin{pmatrix} \vec{u}_m \\ \vec{v}_m \end{pmatrix} \qquad m = 1 \ldots N \text{ and } E_m \geq 0$$

The BdG problem has particle-hole symmetry – for every $E_m > 0$ there is a corresponding $-E_m < 0$ solution

$$\begin{pmatrix} H^0 & \Delta \\ \Delta^\dagger & -H^{0T} \end{pmatrix} \begin{pmatrix} \vec{v}_m^{\,*} \\ \vec{u}_m^{\,*} \end{pmatrix} = -E_m \begin{pmatrix} \vec{v}_m^{\,*} \\ \vec{u}_m^{\,*} \end{pmatrix}$$

Consider translation-inv. problem with $\epsilon_k = \frac{k^2}{2m} - \mu$ ; $\hat{\Delta} = \Delta$ $= \text{const}$



SC opens an energy gap in the fermionic spectrum, the Fermi surface is destroyed, cannot use the Landau Fermi liquid theory.

Use now a linear Bogoliubov transform

$$\mathcal{U}^{-1} H_{BdG} \mathcal{U} = \begin{pmatrix} \vec{E} & 0 \\ 0 & -E \end{pmatrix}$$

$$\mathcal{U} = \begin{pmatrix} u_{im} & v_{im}^* \\ v_{im} & u_{im}^* \end{pmatrix}$$

$$\hat{H}_{MF} = \frac{1}{2} \hat{\Psi}^+ \mathcal{U} \begin{pmatrix} \vec{E} & 0 \\ 0 & -E \end{pmatrix} \underbrace{\mathcal{U}^{-1} \Psi}_{B = \begin{pmatrix} b_m \\ b_m^+ \end{pmatrix}} + \dots$$

$$= \sum_{m=1}^{N} E_m \hat{b}_m^+ b_m - \frac{1}{2} \sum_{m=1}^{N} E_m + \frac{1}{2} \operatorname{tr} H^0 + \frac{1}{g} \Delta^+ \Delta$$

sums only over positive spectrum

$b_m^+$ – creation operator of fermionic quasiparticle with $E_m > 0$

BCS vacuum: $\hat{b}_m |BCS\rangle = 0$ for any $m$

$$|BCS\rangle = \prod_{r=1}^{N} \hat{b}_r \underbrace{|0\rangle}_{\text{Fock vacuum}}$$

by anticommutation relations $\{b_m, b_r\} = 0$ this state is annihilated by all $b_m$.

Expectation values: $\mathcal{O} = \hat{\psi}_i^+ \mathcal{O}_{ij} \psi_j$

$$\langle \hat{\mathcal{O}} \rangle_{BCS} = \langle BCS | \hat{\psi}_i^+ \mathcal{O}_{ij} \hat{\psi}_j | BCS \rangle =$$

$$= \langle BCS | ( v_{im} \hat{b}_m + u_{im}^* b_m^+ ) \, \mathcal{O}_{ij} ( u_{jn} b_n + v_{jn}^* b_n^+ ) | BCS \rangle$$

$$= v_{im} \mathcal{O}_{ij} v_{jm}^*$$