Spontaneous symmetry breaking in fermionic superfluids

$$|BCS\rangle = \exp\Big(\underbrace{\sum_k \varphi_k \psi^\dagger_{k\uparrow} \psi^\dagger_{-k\downarrow}}_{\Lambda^\dagger}\Big)|0\rangle$$

$\Lambda^\dagger \to$ pair creation operator

$$= \sum_n \frac{1}{n!} \underbrace{(\Lambda^\dagger)^n |0\rangle}_{|n\rangle}$$

Transformations under the global particle number $U(1)$ transform

$$\psi^\dagger_{k\sigma} \to e^{i\alpha} \psi^\dagger_{k\sigma}$$

$$\Delta \to e^{-2i\alpha} \Delta$$

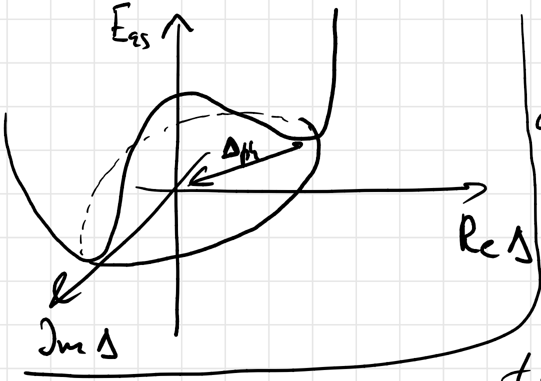
inequivalent states

$$|BCS\rangle \to |BCS\rangle_\alpha = \sum_n \frac{1}{n!} e^{2in\alpha} |n\rangle$$

$\alpha \in (0, \pi) \to$ continuous set of $|BCS\rangle_\alpha$ states

Since E_{BCS} depends only on $|\Delta|$, all states $|BCS\rangle_\alpha$ are degenerate in energy

Superfluids exhibit spontaneous symmetry breaking (SSB) of $U(1)$ symmetry



The microscopic theory of fermions is <u>invariant</u> under global $U(1)$ transformations, but the paired BCS ground state has continuous degeneracy and choses <u>one particular realization</u>.

Δ is called the SC order parameter.

It costs no energy to change the phase of Δ globally.

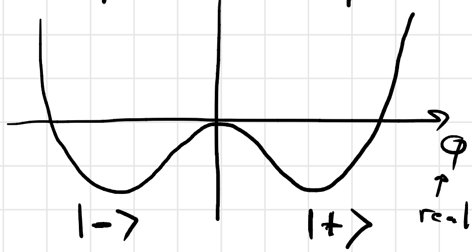
<u>Question:</u> But will the system be in $|BCS\rangle_\alpha$ with some fixed α or in the combination $\sum_\alpha |BCS\rangle_\alpha = |0\rangle$ which is invariant under a $U(1)$ transformation?

To clarify, consider a simpler example

SSB of Ising

$\varphi \to -\varphi$ symmetry

Z_2 symmetry



2d Ising model : $H = -J\sum_{\langle i,j\rangle} Z_i Z_j - h\sum X$ transverse field

$$|+\rangle = \begin{matrix}\uparrow & \uparrow & \uparrow\\ \uparrow & \uparrow & \uparrow\\ \uparrow & \uparrow & \uparrow\end{matrix} \qquad |-\rangle = \begin{matrix}\downarrow & \downarrow & \downarrow\\ \downarrow & \downarrow & \downarrow\\ \downarrow & \downarrow & \downarrow\end{matrix}$$

Spontaneous symmetry breaking

$Z_2 : Q = \prod X$

In the presence of external field, $|+\rangle$ can tunnel into $|-\rangle$, and vice versa, with a finite amplitude.

In the $|+\rangle, |-\rangle$ basis

$$H = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

Z_2-sym. Hamiltonian

which is diagonalized by $|\theta\rangle = |+\rangle - |-\rangle$

$|\mathcal{E}\rangle = |+\rangle + |-\rangle$

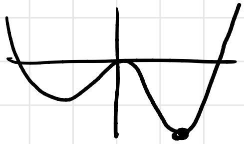
with energies $E_{\theta,\mathcal{E}} = a \mp b$

Actual energy eigenstates $|\theta\rangle, |\mathcal{E}\rangle$ are invariant under the Ising $\phi \to -\phi$ symmetry!

<u>However</u>, $b \sim \exp(-\# L^{\#}) \ll a$ so $|\theta\rangle$ and $|\mathcal{E}\rangle$ are very close to each other in energy.

$\Delta H = \sum Z$

Now switch on a small perturbation which breaks $\phi \to -\phi$ symmetry, the diagonal elements will change substantially (compared to b) and the system will quickly close one of the states



we conclude that in $\Omega \to \infty$ systems the ground state is <u>extremely</u> sensitive to external perturbations that fix the vacuum into one particular sym. broken state. Crucially, if we take the perturbation in a large system to zero now, it takes <u>very long</u> time to tunnel into the symmetric combination (absolute ground state) if $\Omega \to \infty$.

Similar arguments for $U(1)$ SSB — domain walls in XY models.

<u>Conclusion</u>: In large systems (superfluids, magnets, etc) small perturbations <u>practically</u> pin the vacuum (SSB) and the symmetric linear combination is not realized.

Back to the paired $|BCS\rangle$ state:

$$\hat{N}\, |BCS\rangle_\alpha = \sum_n \frac{1}{n!}\, 2n\, e^{2in\alpha} |n\rangle = -i\partial_\alpha |BCS\rangle_\alpha$$

This resembles $p = -i\partial_x$ and thus

$$[\hat{\alpha}, \hat{N}] = i \Rightarrow \Delta\alpha \, \Delta N \gtrsim 1$$

States with small uncertainty of the phase α, have large uncertainty in the particle number.

What is the low-energy theory of a superfluid? Twisting the phase α locally costs little energy $\rightarrow$ it is the low-energy degree of freedom.

Can be seen mathematically from the Ginzburg-Landau free energy

φ - $U(1)$ order parameter,

$$f_{GL} = \underbrace{\frac{\hbar^2}{2M} |\nabla \varphi|^2}_{\text{gradient energy}} + \underbrace{r |\varphi|^2 + \frac{u}{2} |\varphi|^4}_{\substack{r<0 \quad \text{SSB state} \rightarrow \text{mexican hat} \\ r>0 \quad \text{normal state}}}$$

Write the order parameter

$$\varphi = |\varphi| e^{i\theta}$$

$$f_{GL} = \underbrace{\frac{\hbar^2}{2M}|\varphi|^2(\nabla\theta)^2}_{\text{phase part}} + \underbrace{\left\{\frac{\hbar^2}{2M}(\nabla|\varphi|)^2 + r|\varphi|^2 + \frac{u}{2}|\varphi|^4\right\}}_{\text{amplitude part}}$$

no terms that mix θ and $|\varphi|$

Amplitude excitations are gapped, while phase excitations are not. If we neglect the amplitude part

$$f_{GL} = \frac{\hbar^2}{2M}|\varphi|^2(\nabla\theta)^2 \rightsquigarrow \text{kinetic energy of fluid}$$

$$\vec{J} = -i\frac{\hbar}{2M}\left(\varphi^*\vec{\nabla}\varphi - \vec{\nabla}\varphi^*\varphi\right)$$

$$= \frac{\hbar}{M}|\varphi|^2\vec{\nabla}\theta = n_s\vec{v}_s$$

where $n_s = |\varphi|^2$ and $\vec{v}_s = \frac{\hbar}{M}\vec{\nabla}\theta$

and thus $f_{GL} = \frac{M n_s \vec{v}_s^{\,2}}{2}$

Cooper pairs form a coherent state and in superflow move in perfect harmony together, this is the reason why there is no dissipation/viscosity in superflow.

It costs a finite energy to bend the superfluid phase in real space. The cost goes to zero in the $q \to 0$ limit.

θ is an example of a <u>Goldstone boson</u>

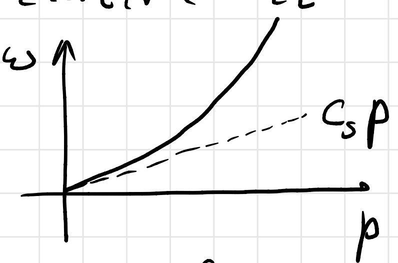
$$\mathcal{L} = \pi \partial_t \theta - \mathcal{H} = \frac{\rho_s}{2c_s^2}(\partial_t \theta)^2 - \frac{\rho_s}{2}(\partial_i \theta)^2$$

massless scalar

where c_s – speed of sound

1) Due to SSB, θ appears only with derivatives, its mass term is prohibited

2) It is gapless and has linear dispersion relation at small momenta



Sound mode in the superfluid

<u>Nambu-Goldstone theorem</u>: In systems with Lorentz symmetry every <u>continuous</u> spontaneously broken symmetry gives rise to one <u>linearly disspersive</u> mode (1960 Nambu, 1961 Goldstone)

In condensed matter systems Lorentz invariance is violated $\Rightarrow$ Goldstones with softer dispersions are possible, e.g. magnons in ferromagnet $\omega \sim p^2$

In 2011 Watanabe, Brauer, Murayama worked out the Lorentz non-invariant version of NG theorem. PRL 108 251602 2012

Nambu-Goldstone modes are abundant in physics

Condensed matter – phonons in crystals
Bogoliubov superfluid modes
magnons in ferromagnets

Particle physics – pions