

Problem set 2: Propagator of a two-dimensional particle in a constant magnetic field

Due date: Nov 13

It was discovered by Landau in 1930 that in quantum mechanics the energy spectrum of a two-dimensional particle of unit charge and mass m embedded into a constant magnetic field B forms an equidistant tower of degenerate levels $E_n = \omega(n + 1/2)$, where we set $\hbar = 1$ and introduced the cyclotron frequency $\omega = B/m$. The degeneracy of one Landau level N_{LL} is fixed by the total magnetic flux piercing the system, $N_{LL} = BS/(2\pi)$ with S being the area.

In this problem set you will calculate the propagator of a particle in a constant magnetic field with the help of the Feynman path integral. We start from the classical action

$$S = \int_0^t ds \left(\frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \dot{x}_i A_i \right) = \int_0^t ds \left(\frac{m}{2} (\dot{x}^2 + \dot{y}^2) - \frac{B}{2} \epsilon_{ij} \dot{x}_i x_j \right). \quad (1)$$

As discussed in the lecture, since the action is quadratic, the propagator is given by

$$K(x', y', x'', y''; t) = \int \mathcal{D}x \mathcal{D}y e^{iS[x, y]} = \mathcal{N} e^{iS[x_{cl}, y_{cl}]}, \quad (2)$$

where in the last expression the action is evaluated on the classical trajectory that satisfies the proper boundary conditions $\mathbf{r}(s=0) = (x', y')$ and $\mathbf{r}(s=t) = (x'', y'')$. For this problem you can take as given that for this problem $\mathcal{N} = B/(4\pi i \sin \frac{\omega t}{2})$.

1. From the action (1) derive classical equations of motion and determine which trajectory the particle follows. Convince yourself that the expression

$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} + \frac{\sin \frac{\omega}{2} s}{\sin \frac{\omega}{2} t} \begin{pmatrix} \cos \frac{\omega}{2} (s-t) & \sin \frac{\omega}{2} (s-t) \\ -\sin \frac{\omega}{2} (s-t) & \cos \frac{\omega}{2} (s-t) \end{pmatrix} \begin{pmatrix} x'' - x' \\ y'' - y' \end{pmatrix} \quad (3)$$

solves the classical equation of motion and satisfies the boundary conditions.

2. Evaluate the action on the classical solution (3) and show (on paper or using computer) that it is given by

$$S[x_{cl}, y_{cl}] = \frac{m\omega}{4} \cot \frac{\omega t}{2} ((x'' - x')^2 + (y'' - y')^2) + \frac{m\omega}{2} (x' y'' - x'' y'). \quad (4)$$

Why does this action diverge at integer multiples of the cyclotron period?

3. Perform analytic continuation of the problem to the imaginary time $t \rightarrow -i\tau$. First show that the Euclidean action is

$$S_E = \int_0^\tau ds \left(\frac{m}{2} (\dot{x}^2 + \dot{y}^2) + i \frac{B}{2} \epsilon_{ij} \dot{x}_i x_j \right). \quad (5)$$

Compute the Euclidean propagator $K_E(x', y', x'', y''; \tau)$.

4. Given the Euclidean propagator $K_E(x', y', x'', y''; \tau)$, compute the canonical partition function of a particle in a magnetic field

$$Z = \int_S dx dy K(x, y, x, y; \beta), \quad (6)$$

where β is the inverse of the temperature T . Compare your result with the partition function computed by summing over Landau levels

$$Z = \sum_{LL} e^{-\beta E_{LLL}}, \quad (7)$$

where the sum is taken over all quantum states (do not forget about Landau level degeneracies).

5. Take the lowest Landau level (LLL) limit $\omega \rightarrow \infty$ by formally sending $m \rightarrow 0$ and keeping $B = \text{const}$. The resulting Euclidean propagator $K_{E,LLL}(x', y', x'', y''; \tau)$ exhibits Gaussian decay. Determine the length scale of this decay and interpret it. Do you know why the prefactor \mathcal{N} goes to zero in the LLL limit?