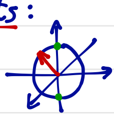
17 Topological Quantum Computation

17.1 Quantum Computer

Classical computer use bits : 0, 1 and logical operations (Or / xor / and)

Quantum computer use qubits :

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



more freedom → Comp. power
Manipulations using unitary transformations
SU(N) for N qubits

Universal quantum computer:
Able to simulate to arbitrary precision all SU(N)

Decomposition into:

Single qubit rotations $|\psi\rangle \to U|\psi\rangle$, $U = e^{-i(\theta + \alpha \vec{n}\vec{\sigma})}$

Sufficient to have two rotations that are dense in SU(2)

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} -e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$$

(Hadamard)

CNOT operation: Flips the second qubit if and only if the first qubit is $|1\rangle$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{array}{l} |0\rangle|0\rangle \\ |0\rangle|1\rangle \\ |1\rangle|0\rangle \\ |1\rangle|1\rangle \end{array}$$

$\rightsquigarrow$ Generates entanglement

Why quantum computer?

- Shor algorithm: exponential speed up of prime factor decomposition (decryption RSA)
- Grover algorithm: Search data bases efficiently $N \to \sqrt{N}$
- Deutsch algorithm: Figure out if $f(0) = f(1)$ [with $f(\cdot) \in \{0,1\}$] twice as fast as classical computer

1.) Prepare $|0\rangle|1\rangle$

2.) Apply Hadamard gate $\rightsquigarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \cdot \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

3.) Apply gate $G|i\rangle|j\rangle := |i\rangle|j + f(i)\rangle$ ['+' mod 2] to the above $\rightsquigarrow (-1)^{f(0)} \frac{1}{2}\left(|0\rangle + (-1)^{f(0)+f(1)}|1\rangle\right)(|0\rangle - |1\rangle)$

4.) Apply H again to the output above (ignore phase and last bit)
$\rightsquigarrow \frac{1}{2}\left[(1 + (-1)^{f(0)+f(1)}|0\rangle\right]|0\rangle + \left[1 - (-1)^{f(0)+f(1)}\right]|1\rangle$

5.) We can read off the result by measuring:
$|0\rangle: f(0) = f(1)$ $\quad$ $|1\rangle: f(0) \neq f(1)$

Local errors (thermal fluctuations / coupling to environment)

Proposal by Kitaev:
Build a topological quantum computer

Robust to local perturbations as small perturbations do not change the topology

Non-Abelian anyons can realize a universal quantum computer!

17.2 Non-abelian anyons

$$\Psi_a(x_1, x_2, \dots) = \sum_b M_{ab} \Psi_b(x_2, x_1, \dots)$$

degenerate GS's

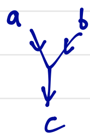
Main ingredients for a theory of anyons:

(i) List of types of anyons

$\{1, a, b, \dots\}$

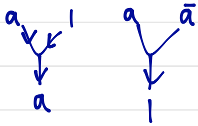
(ii) Rules for fusing/splitting pairs of anyons

$$a \times b = \sum_c N_{ab}^c \, c$$



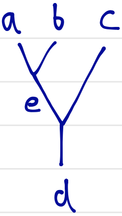
Abelian anyons $\Leftrightarrow$ unique fusion $(a \times b = c)$

Vacuum 1 fuses trivially
$\bar{a}$ fuses with a to 1

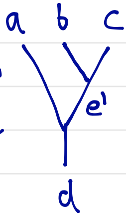


Associativity of fusion : F-matrix

$$(a \times b) \times c = a \times (b \times c)$$



$$= \sum_{e'} [F^{d}_{abc}]_{e}^{e'}$$



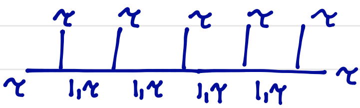
Fusion space : Space spanned by fusing N anyons

Example: Fibonacci anyons

Two QP : 1, τ

$1 \times 1 = 1 \quad (N_{11}^{1} = 1)$
$1 \times \tau = \tau \times 1 = \tau \quad (N_{1\tau}^{\tau} = N_{\tau 1}^{1} = 1)$
$\tau \times \tau = 1 + \tau \quad (N_{\tau\tau}^{1} = N_{\tau\tau}^{\tau} = 1)$
(like fusing two $S = 1/2$ to $S = 0, 1$)

How many ways are there to fuse?



$1 : 1$

$\tau : 1$

$\tau \times \tau = 1 + \tau \quad : 2$

$\tau \times \tau \times \tau = \tau(1+\tau) = \tau + 1 + \tau \; : 3$

$\tau \times \tau \times \tau \times \tau = \tau(2\tau+1) = 2(\tau+1) + \tau \; : \; 5$

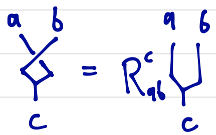
Fibonacci series! Number of states grows as

$$\sim \left(\frac{1+\sqrt{5}}{2}\right)^N =: d^N, \quad d = \frac{1+\sqrt{5}}{2}$$

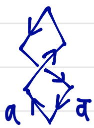
Asymtotic growth: Quantum dimension d

Abelian anyons have $d = 1$

(iii) Rules for braiding anyons



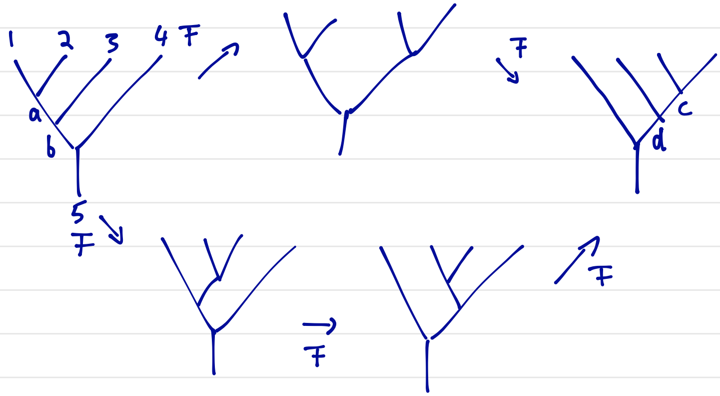
The R-matrix determines top. spin: phase when rotating particle by 2π



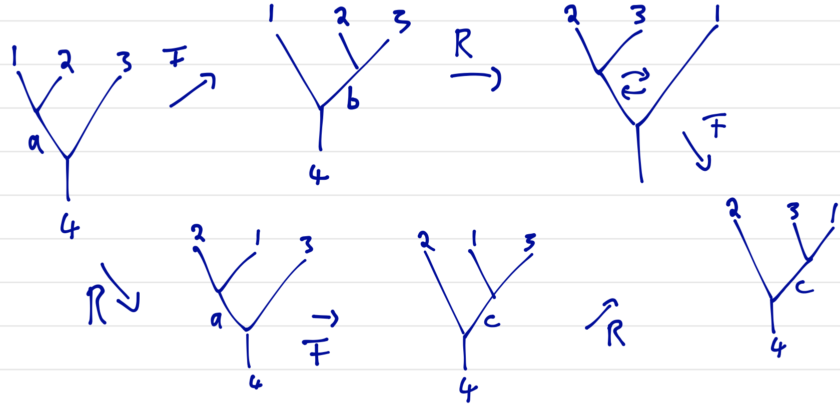
Example: Fibonacci anyons $R^0_{11} = e^{-i4\pi/5}$, $R^1_{11} = e^{i3\pi/5}$

Fusion and braiding have to be chosen consistently: Pentagon & hexagon equation

Pentagon Equation $\rightsquigarrow$ Consistency of Fusion alone



Hexagon equation $\rightsquigarrow$ Consistency of Fusion and braiding

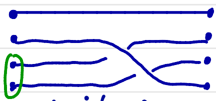


Design your own anyon model!

17.3 Computation with anyons

Braiding of anyons to implement gates

qbit



Measurement

Unitaries

Universal TQC can be realized using Fibonacci anyons