15 Toric code and topological order

Properties of topological order in 2D:

(1) Quantum phase of matter that is not characterized by sym. breaking and it does not require any sym. protection.

(2) Gives rise to emergent anyonic excitations.

(3) Topological ground state degeneracy.

(4) Topological entanglement entropy

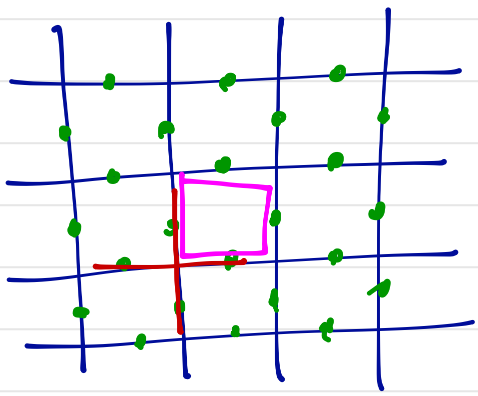
15.1 Toric code

Exactly solvable quantum spin model with $\mathbb{Z}_2$ topological order.

Square lattice with spins on edges

$$H_T = -\sum_{+} A_+ - \sum_{\square} B_\square$$

$$A_s = \prod_{j\in +} \sigma_j^z \,, \quad B_p = \prod_{j\in\square} \sigma_j^x$$



All terms commute! Diagonalize terms simultaneously.

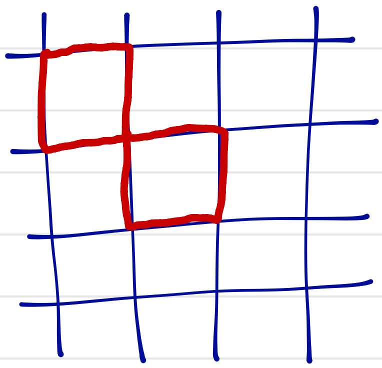
Choose a basis of σ^z eigenstates $s_j = \pm 1$

$$A_+ |\psi\rangle = |\psi\rangle$$

$$\rightsquigarrow \quad |\psi\rangle = \sum_{\{s \,|\, \forall_+ : \prod_{j\in +} s_j = 1\}} c_s |s\rangle$$

Representation as loop model

$|\uparrow\rangle \mathrel{\hat{=}} |\,\vdots\,\rangle$, $|\downarrow\rangle \mathrel{\hat{=}} |\,|\,\rangle$

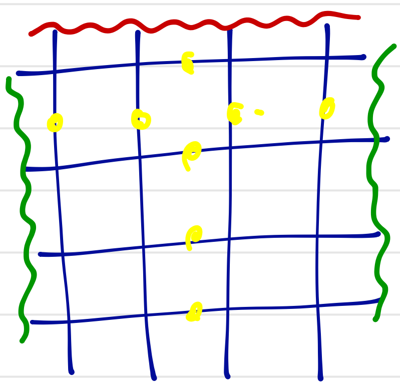


$$B_\Box |\psi\rangle = |\psi\rangle$$

$\rightsquigarrow$ all c_s must be equal for connected states

Torus: H conserves

$W_\gamma = \prod_{j\in\gamma} s_j$, $\gamma = \gamma_x, \gamma_y$



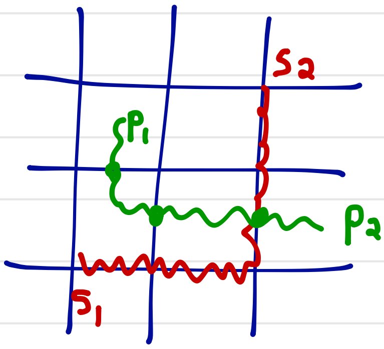
GS is fourfold degenerate:

$|\psi_{00}\rangle, |\psi_{10}\rangle, |\psi_{01}\rangle, |\psi_{11}\rangle$

$\rightsquigarrow$ Topological degeneracy

15.2 Anyonic excitations

electric charges:



$$W_\ell^e = \prod_{j\in\ell} \sigma_j^x$$

$$[W_\ell^e, B_\Box] = 0$$

$[W_\ell^e, A_+] = 0$, except at end points s_1, s_2 where one edge overlaps

$\rightsquigarrow |\psi^e_{s_1 s_2}\rangle = W_\ell^e |\psi_0\rangle$, $\Delta E = 2$

magnetic vortices: $W_\ell^m = \prod_{j\in\ell} \sigma_j^z$

Analogous contruction $\rightsquigarrow |\psi^m_{p_1 p_2}\rangle = W_\ell^m |\psi_0\rangle$, $\Delta E = 2$

Anyonic excitations are the defining property of top. order!

Fusion:

$a \sim b$ if a can be transformed to b by local operator

$e \times e = 1$

$\mathbb{Z}_2$ liquid has four particles : $1, e, m, \varepsilon = e \times m$

$e \times e = 1$, $m \times m = 1$, $\varepsilon \times \varepsilon = 1$

$e \times m = \varepsilon$, $m \times \varepsilon = e$, $e \times \varepsilon = m$

exchange statistics:

Information about exchange statistics

contained in U-Matrix

$$U_{aa} = (1, 1, 1, -1) \qquad [a = 1, e, m, \varepsilon]$$

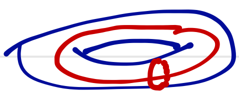
↑ Fermion!

mutual statistics:

$$S_{ab} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Mutual statistics yields top. degeneracy

$$W^{e^{-1}}_{\gamma_x} \cdot W^{m^{-1}}_{\gamma_y} W_{\gamma_x} W_{\gamma_y} = -1$$



Four such operators ⇝ fourfold degeneracy
(degeneracy = # anyons)

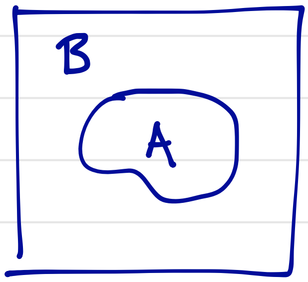
The content of U and S matrices classifies top. ordered phases ~ not known to be a complete classification!

15.3 Topological entanglement

Recall: Entanglement entropy

Reduced density matrix:

$$\rho^A = \mathrm{Tr}\left[|\psi\rangle\langle\psi|\right]$$



Entanglement entropy:

$$S_A = S_B = -\mathrm{Tr}\,\rho\log\rho = -\sum \rho_\alpha \log \rho_\alpha \quad , \quad \rho|\alpha\rangle = \rho_\alpha|\alpha\rangle$$

Schmidt decomposition $\quad |\psi\rangle = \sum \sqrt{\rho_\alpha}\, |\alpha\rangle_A |\alpha\rangle_B$

Area law in 2D: $S \sim L$ (length of the cut)

Corrections to the area law in topologically ordered states.

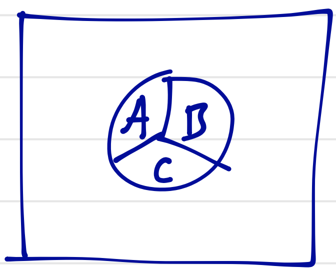
$$S = \alpha L - \gamma_{top} \qquad \text{(Levin and Wen, Kitaev-Preskill)}$$

$$\boxed{\gamma = \log D}$$

D: total quantum dimension, Abelian models: $D = \sqrt{\#QP}$

Conveniently calculated by

$$-\gamma = S_A + S_B + S_C - S_{BC} - S_{AC} - S_{AB} + S_{ABC}$$



Note that TEE is the same for Renyi entropies

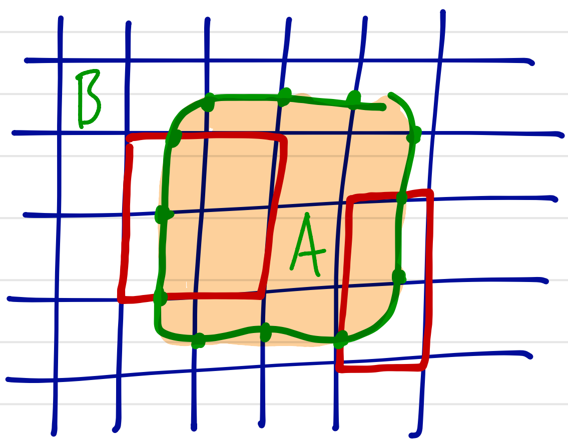
$$S_n = \frac{1}{1-n} \log\left(\mathrm{Tr}\, \rho^n\right)$$

Renyi entropies can be obtained from MC and potentially measured in cold atom experiments.

Example: Toric code

$$|\psi\rangle = \sum \sqrt{\rho_{s_1 \dots s_L}} \; |s_1 \dots s_L\rangle_A |s_1 \dots s_L\rangle_B$$

$\prod_{j=1}^{L} s_j = +1$ as all loops are closed



⇝ Lowers entropy by 1 bit

$$S = -\sum \frac{1}{2^{L-1}} \cdot \log \frac{1}{2^{L-1}}$$

$$= \log 2^{L-1} = L \cdot \log 2 - \underbrace{\log 2}_{=\gamma}$$