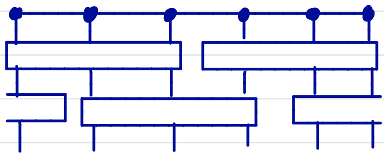
13 Symmetry Protected Topological phases (SPT)

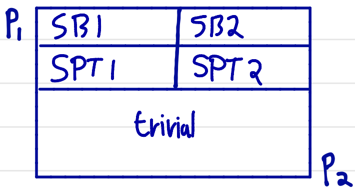
Quantum phases of matter in 1D (gap, $T=0$)

Two states are in the same phase iff there exists a local unitary that transforms one into the other:

$$U = \mathcal{T}\, e^{-i\int_0^1 dg\, H(g)}$$



Schematic phase diagram:



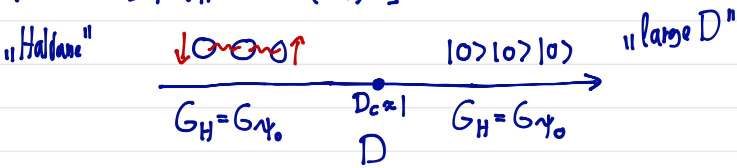
SB $\hat{=}$ Spontaneous sym. breaking (Landau theory).
Trival $\hat{=}$ Adiabatically connected to sym. product state.

SPT $\hat{=}$ Phase of matter is not characterized by SB but instead it is protected by a symmetry.

<u>Example</u>

$S=1$ chain with single ion anisotropy

$$H = \sum \left[\vec{S}_i \vec{S}_{i+1} + D\cdot (S_i^z)^2\right]$$



Symmetries: Translation, $\mathbb{Z}_2 \times U(1) \supseteq \mathbb{Z}_2 \times \mathbb{Z}_2$, TR, Inversion

How to distinguish the phases?

13.1 Bosonic SPT order in 1D

1D SPT stabilized by different symmetries: on-site, TR, inversion,...
Focus on on-site symmetries, e.g. spin flip symmetries:

Projective representations

Operators $u(g)$ form a projective representation (PR) if

$$u(g) \cdot u(h) = e^{i\phi_{gh}} \cdot u(g \cdot h),$$

$\{e^{i\phi_{gh}}\}$ is the factor set (fulfils consistency cond.: Associativity).

[2-cochain $G \times G \to U(1)$]

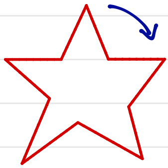
If $\forall gh: \phi_{gh} = 0$, the representation is linear.

Allowing to change the phases $u'(g) = e^{i\alpha} \cdot u(g)$, which PR can be transformed into each other?

$$\phi'_{gh} = \phi_{gh} + \alpha_{gh} - \alpha_g - \alpha_h$$

Equivalent PR belong to the same cohomology class $H^2(G, U(1))$

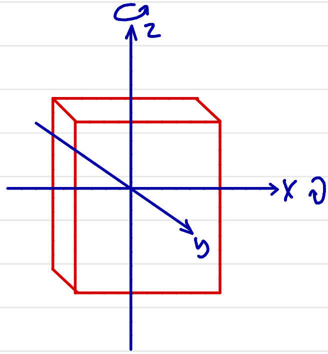
Example $\mathbb{Z}_N : \{1, R, R^2, \dots, R^{N-1}\}$



$R^N = \mathbb{1} \Rightarrow U^N = e^{i\phi}\mathbb{1}$

Redefinition $\tilde{U}_R = e^{-i\phi/N} U_R$ removes the phase : All pr. rep. equivalent

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2 : \{1, R_x, R_z, R_x R_z\}$



$U_x^2 = U_z^2 = 1$

$R_x R_z = R_z R_x \Rightarrow U_x U_z = e^{i\phi} U_z U_x$

Phase $\phi = 0, \pi$ cannot be removed : 2 classes

[Integer/ half integer representation]

Classification of 1D SPT

$|\psi\rangle$ symmetric under $\otimes_j U_j(g)$ with $U_j(g)$ being a linear on-site representation of g:

X dimensional

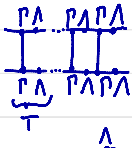
Projective representation

$$\Gamma \text{ with } u \;=\; e^{i\theta}\; V\,\Gamma\,V^{\dagger}, \quad [V, \Lambda] = 0$$

linear representation of $g \in G$

$$\boxed{\text{The projective representation together with the phase } e^{i\theta} \text{ classify different SPT.}}$$

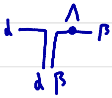
Sketch of a proof:

1.) Combine n transfer matrices $\tilde{T}_{\alpha\alpha',\beta\beta'} =$  .

If $n \gg \xi$, we can write $\tilde{T}_{\alpha\alpha',\beta\beta'} =$  .

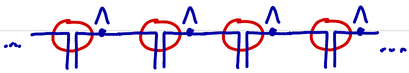
This expression can be decomposed

$$\tilde{T}_{\alpha\alpha',\beta\beta'} = \sum_i \tilde{A}^i_{\alpha\beta} \tilde{A}^{i*}_{\alpha'\beta'}$$

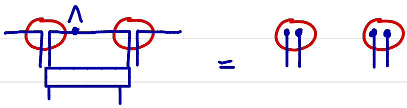
with $\tilde{A}^{(\alpha\beta)}_{\alpha\beta} =$  being the fix-point MPS.

(same entanglement as the original MPS)

2.) Study the fix point MPS:



a) Without symmetry: we can always find a local unitary U that disentangles the state



⇝ Every state can be transformed to a product state

b) With symmetry: Two fix-point MPS $\tilde{A}$ and $\tilde{B}$ can be transformed by local unitary iff both states have projective representations of the same class.

Example: $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry in $S=1$ spin chain

Sufficient to look at two generators, e.g. R_x, R_z.

On-site representation in terms of $S=1$ operators:

$$u_x = e^{i\pi S_x}, \quad u_z = e^{i\pi S_z} \quad \text{(linear representations)}$$

Thus the MPS transforms as

$$\sum_k [u_x]_{jk}\, \Gamma_k = e^{i\theta_x}\, V_x \Gamma_k V_x^\dagger$$

$$R_{x/z}^2 = 1 \Rightarrow \theta_{x/z} = 0, \pi$$

V_x, V_z can be linear or projective representions of $\mathbb{Z}_2\times\mathbb{Z}_2$, i.e., $V_x \cdot V_z = \pm V_z V_x$.

AKLT: $\Gamma^{-1} = \sqrt{\frac{2}{3}}\begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad \Gamma^0 = -\frac{1}{\sqrt{3}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \quad \Gamma^{+1} = -\sqrt{\frac{2}{3}}\begin{pmatrix}0&0\\1&0\end{pmatrix}$

$$u_z = \begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix} \leadsto V_z = \sigma_z$$

$$\left[\text{e.g., } \begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-1\\0&0\end{pmatrix} = \begin{pmatrix}0&-1\\0&0\end{pmatrix}\right]$$

$$u_x = \begin{pmatrix}&&-1\\&-1&\\-1&&\end{pmatrix} \leadsto V_x = \sigma_x$$

$$\leadsto V_x V_z = \underline{\underline{-}}\, V_z V_x$$

$|0\rangle|0\rangle$: $V_x = V_z = \mathbb{1}$

$$\leadsto V_x V_z = \underline{\underline{+}}\, V_z V_x$$

- AKLT phase has degeneracies in the ES since
  Λ commutes with V_x, V_z and $V_x V_z = -V_z V_x$.

$$\left[\Lambda|\phi_n\rangle = \Lambda_n|\phi_n\rangle, \quad \Lambda \underbrace{V_x|\phi_n\rangle}_{|\phi_n'\rangle} = \Lambda_n \cdot V_x|\phi_n\rangle \text{ and } \langle\phi_n'|\phi_n\rangle = 0 \right]$$

- AKLT can be generalized to higher spin S

  even S : fractionalization into integer spin $\Rightarrow$ „trivial"
  odd S : fractionalization into half-integer spin $\Rightarrow$ SPTP

Inversion symmetry

SPT can be protected by inversion symmetry

$$\begin{array}{cccc|ccc}
\circ & \circ & \circ & & \circ & \circ & \circ \\
-3 & -2 & -1 & & 0 & 1 & 2
\end{array} \quad : \quad i = -i-1$$

MPS transforms as $\Gamma \to \Gamma^T$: $\Gamma^T = \pm V \Gamma V^\dagger$ and $V = \pm V^T$

Time reversal symmetry

MPS transforms as $\Gamma \to \Gamma^*$: $\Gamma^* = \pm V \cdot \Gamma V^\dagger$ and $V = \pm V^T$

13.2 Signatures of SPT phases

How to detect an SPT? No local order parameter!

Edge modes

Finite segment of an SPT has sym. protected, gapless edge modes.

Example : AKLT

$\rightsquigarrow$ 4-fold degeneracy of the ground state
(eg. Kramer's degeneracy on both ends)

String order parameter

Suppose the state $|\Psi\rangle$ is invariant under $g, h \in G$, such that $\left[\bigotimes_j u^j_{g/h}\right] |\Psi\rangle = e^{i\phi} |\Psi\rangle$.

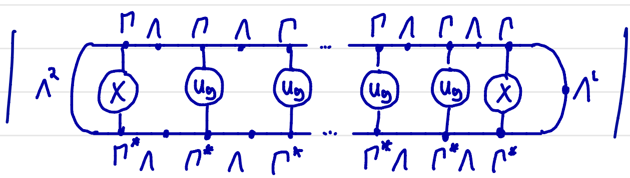
Define the string order parameter as

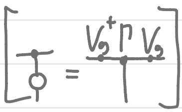
$$S(g,X) = \left| \lim_{|i-j| \to \infty} \langle \Psi | X_i \left[\bigotimes_{k=2}^{n-1} u_g^k \right] X_j \, | \Psi \rangle \right|$$

On the first glance : S should be non-zero for any state that is symmetric under g!

Turns out to be more subtle! If X is choosen approriately, S can distinguish SPT phases. Use the MPS representation to understand S:

$S(g,X) =$





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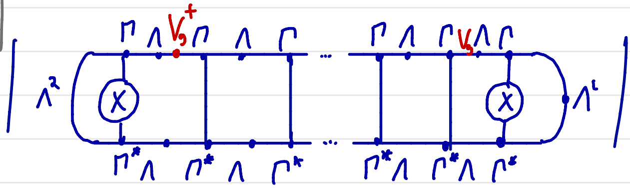
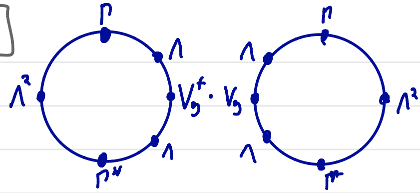
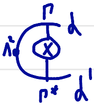


[Canon. Form!]

$=$ 

Suppose that $u_g u_h = u_h u_g$ and $V_h V_g = e^{i\phi} V_g V_h$

Choose X to have a particular quantum number under h : $u_h \cdot X \cdot u_h^\dagger = e^{i\sigma} X$.

The matrix $X^L_{dd'} =$ 

transforms the same way under V_h $[V_h X^L V_h^\dagger = e^{i\sigma} X^L]$.

(similarly for X^R on the right)

It follows that $tr[\Lambda X^L \Lambda V_g^\dagger] = tr[V_h \Lambda X^L \Lambda V_g^\dagger V_h^\dagger]$

$$= e^{i(\sigma-\phi)} \cdot tr[\Lambda X^L \Lambda V_g^\dagger]$$

$\rightsquigarrow$ Selection rule : String order vanishes if $\phi \neq \sigma$!

Vanishing string order in a symmetric phase is what is surprising!

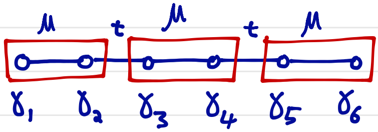
Example: $S=1$ chain with $\mathbb{Z}_2 \times \mathbb{Z}_2$ sym.

$$S(g,X) = \left| \lim_{|i-j|\to\infty} \langle \Psi | S_i^z \left[\bigotimes_{k=2}^{n-1} e^{i\pi S_k^z} \right] S_j^z | \Psi \rangle \right|$$

$$H = \sum \left[\vec{S}_i \vec{S}_{i+1} + D \cdot (S_i^z)^2 \right]$$

13.2 Majorana chain revisited

Dimerized chain of Majorana Fermions:



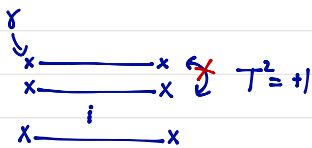
Described by the Hamiltonian

$$H = i\sum_{j=1}^{L-1} t\,\gamma_{2j}\gamma_{2j+1} + i\sum_{j=1}^{L} \mu\,\gamma_{2j-1}\gamma_{2j}$$

which commutes with $P = \prod_{j=1}^{L}\left[-i\gamma_{2j-1}\gamma_{2j}\right]$ (Parity symmetry).

Hamiltonian is real $\mathcal{T}^2 = +1$ ⇝ Class BDI with $\mathbb{Z}$ top. phases

Stacking of chains:



Majorana modes cannot be coupled by quadratic terms as they violate $\mathcal{T}^2 = 1$!

What if we allow quartic terms (e.g. $\gamma\gamma\gamma\gamma$)?

Coupling 8 Majorana fermions with quartic term can "gap out" the edge mode ⇝ trivial phase

Thus $\mathbb{Z} \to \mathbb{Z}_8$ in presence of interactions.

Different SPTs classified by how symmetries act on the edges:

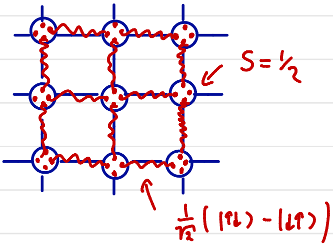
- $P = P_L \cdot P_R$, $P_L \cdot P_R = e^{-i\mu} P_R P_L$, $\mu = 0, \pi$
- The way $\mathcal{T}$ acts on the edges, $\kappa = 0, \pi$
- Commutation relation of P and $\mathcal{T}$, $\phi = 0, \pi$

13.3 Bosonic SPT in 2D

Generalization of AKLT to 2D:

S=2 state on the square lattice.

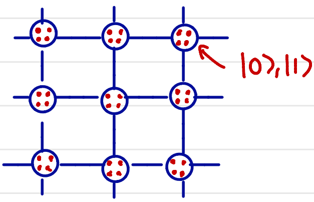
A segment cut out of 2D plane:



⇝ $S=\frac{1}{2}$ degrees of freedom at the edge
⇝ Can be removed if translation symmetry is broken (dimerization)
⇝ SPT phase only if translation symmetry is preserved!

CZX-model: $\mathbb{Z}_2$ sym. protected SPT-phase

SPT that does not require translation symmetry.
Consider a square lattice with four Ising spins per site:



Onsite symmetry

Symmetry group generated by

$$U_{CZX} = U_X U_{CZ}.$$

with $U_X = \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x$

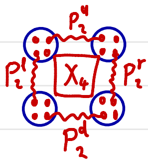
$$U_{CZ} = CZ_{12}\, CZ_{23}\, CZ_{34}\, CZ_{41},$$

$$CZ = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|$$

$\rightsquigarrow$ $\mathbb{Z}_2$ group $[U_{CZX}^2 = 1]$

Hamiltonian contains local terms around plaquettes

$$H = \sum_p H_p \ , \quad H_p = -X_4 \cdot P_2^u P_2^d P_2^l P_2^r$$

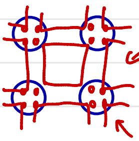


with $X_4 = |0000\rangle\langle 1111| + |1111\rangle\langle 0000|$ and

$$P_2 = |00\rangle\langle 00| + |11\rangle\langle 11|$$

All term commute $\rightsquigarrow$ $|\psi_0\rangle = \bigotimes_p \left[\,|0000\rangle + |1111\rangle\right]$ unique GS, CZX sym.

Non trivial character at the boundary: Study reduced density matrix ρ for a block



$\rho_2 = |00\rangle\langle 00| + |11\rangle\langle 11|$

$\rho_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$

Only degrees of freedom at the boundary are free.

$\rightsquigarrow$ Effective degrees of freedom $|\tilde{1}\rangle = |11\rangle$, $|\tilde{0}\rangle = |00\rangle$

$(|\tilde{1}\rangle = |1\rangle, |\tilde{0}\rangle = |0\rangle$ at corner$)$

CZX sym. acts non trivially on the effective spins at boundary:

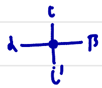
$$\tilde{U}_{CZX} = \prod_j \tilde{\sigma}_j^x \cdot \prod_j \tilde{CZ}_{j,j+1} .$$

Special symmetry:

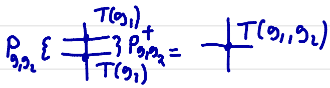
(1) Not an on-site symmetry
(independent of how we group the sites)

(2) No term with CZX symmetry can yield a unique, gap state at the boundary ⇝ SPT phase

Proof: Express $\tilde{U}_{CZX}$ as matrix product unitary (MPU) $T^{ii'}_{\alpha\beta}(g) =$

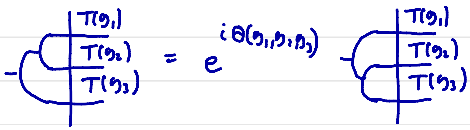


Combined symmetry operations:



⇝ $P_{g_1 g_2}$ defined up to an arbitrary phase factor

Non-trivial phase factors when combining three symmetry



(same proj. on the right)

⇝ $\Theta(g_1, g_2, g_3)$ form a 3-cocycle, classified by $H^3(G, U(1))$
⇝ CZX is non-trivial!

No gapped state can have a sym. with non-trivial MPU!
(Assumption that the state can be expressed by MPS leads to contradiction ~ hexagon equation)
see Xie et al. PRB 84, 235141