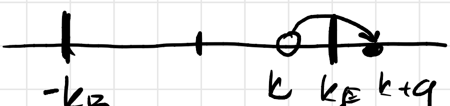
Luttinger liquids

In 1d no single-particle excitations, but necessarily collective excitations

2d or higher 1d

We will see that the Landau Fermi liquid paradigm breaks down in 1d and must be replaced by a different theory.

Particle-hole excitations in 1d:



$$E_{ph}(q) = \epsilon(k+q) - \epsilon(k) = \frac{q^2}{2m} + \frac{kq}{m}$$

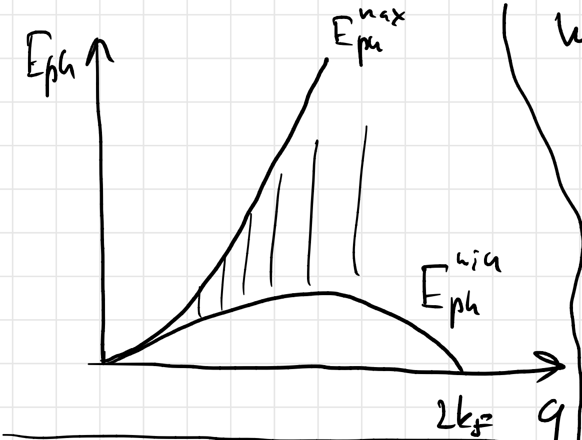
$k \in (-k_F, k_F)$

$k = k_F$

$$E_{ph}^{max} = \epsilon(k_F + q) - \epsilon(k_F) = \frac{q^2}{2m} + \frac{k_F q}{m}$$

$k = k_F - q$

$$E_{ph}^{min} = \epsilon(k_F) - \epsilon(k_F - q) = -\frac{q^2}{2m} + \frac{k_F q}{m}$$



We observe
$$\frac{\Delta E_{ph}(q)}{E_{ph}(q)} \to 0$$
Contrary to higher dimensions, in 1d particle-hole excitations become sharp

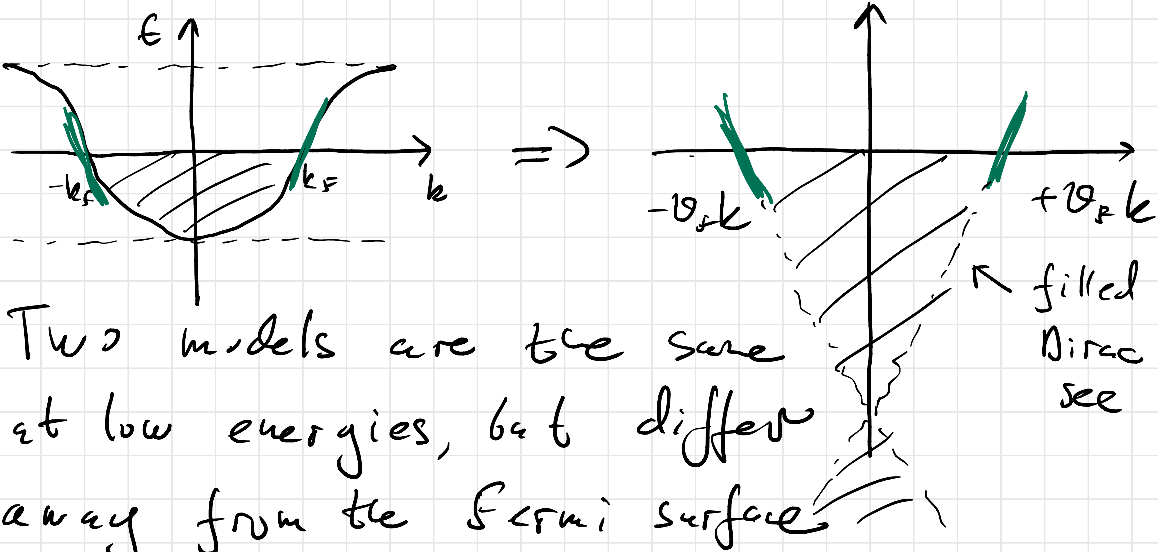
$$\boxed{E_{ph} = \left.\frac{\partial \epsilon}{\partial k}\right|_{k_F} \cdot q}$$ as $q \to 0$

P-h excitation is bosonic, first glimple of ~~bosonization~~

Here we will mainly follow
T. Giamarchi "Quantum physics in 1d"
see also C. Kane, Boulder Summer School lecture notes on bosonization, 2005

Tomonaga - Luttinger model

Consider an idealized relativistic model



Two models are the same at low energies, but differ away from the Fermi surface

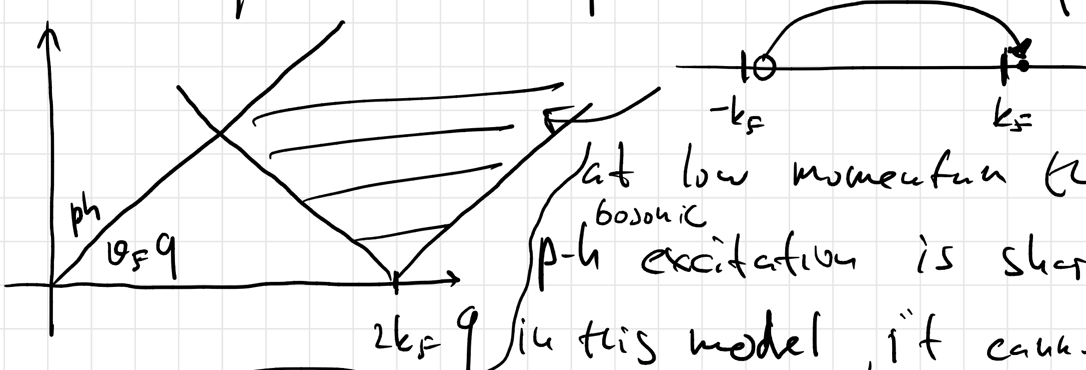
$$H_{TL} = \sum_{\substack{k \\ r=R,L}} v_F (s_r k - k_F)\, c^\dagger_{r,k} c_{r,k}$$

$S_R = +1,\ S_L = -1$

all negative energy states are filled, Dirac see

Particle-hole excitations in TL model:

$$E_{ph}(q) = v_F(k+q) - v_F k = v_F q$$



at low momentum the bosonic p-h excitation is sharp in this model, it cannot decay

The main idea of the Luttinger theory - use bosonic language to describe exc.

density $\rho(x) = c^{\dagger}(x)\, c(x)$

$\uparrow$ fermion

$$\underset{\rho(-q)}{\underset{\parallel}{\rho^{\dagger}(q)}} = \sum_k c^{\dagger}_{k+q} c_k \leftarrow \text{convolution}$$

sums of p-h excitations

Our plan: write $\rho(q)$ in terms of $b_q\, b_q^{\dagger}$

$$\rho^{\dagger}(q) = \#\, b_q + \#\, b_q^{\dagger}$$

and express H_{TL} in terms of $b, b^{\dagger}$

$\rightsquigarrow$ surprisingly the result will be quadratic. In addition, in this language density - density interactions, that are quartic in fermions

$$H_{int} \sim \frac{1}{V} \sum_q V(q)\, \rho(-q)\, \rho(q)$$

$$\sim \frac{1}{V} \sum_q V(q) \left(\#\, b_q + \#\, b_q^{\dagger}\right)^2$$

are also quadratic in terms of $b, b^{\dagger}$ and thus $H_{tot} = H_{TL} + H_{int}$ can be easily diagonalized.

Mathematically, need to treat Dirac see carefully! To avoid infinities introduce

Normal ordering : $:\mathcal{O}:$

move all creation/annihilation left/right

by construction $\langle 0 | :\mathcal{O}: | 0 \rangle = 0$

Dirac see

for $\hat{A}, \hat{B}$ ~ linear combination of creation and distruction operators

$$:\hat{A}\hat{B}: = \hat{A}\hat{B} - \underbrace{\langle 0 | \hat{A}\hat{B} | 0 \rangle}_{\substack{\text{subtraction term} \\ \text{c-number}}}$$

Consider now an example:

$$:\rho_r(x): = :c_r^+(x)\, c_r(x):$$

We go now to Fourier space

$$:\rho_r(x): = \frac{1}{V} \sum_p :\rho_r(p): e^{ip\cdot x}$$

$$\underbrace{:\rho_r^+(p):}_{\text{finite matrix elements}} = \begin{cases} \sum_k c^+_{r,k+p}\, c_{r,k} & \text{if } p \neq 0 \\ \sum_k c^+_{r,k}\, c_{r,k} - \langle 0 | c^+_{r,k} c_{r,k} | 0 \rangle & \text{if } p = 0 \end{cases}$$

infinity after summation

Now we want to compute the commutator

$$[\rho_r^+(p), \rho_{r'}^+(-p')]$$

it is nontrivial only if $r = r'$.

First we do a naive calculation,

$$[\rho_r^+(p), \rho_r^+(-p')] = \sum_{k_1,k_2} [c^+_{k_1+p} c_{k_1}, c^+_{k_2-p'} c_{k_2}]$$

$$= \sum_{k_1,k_2} \left(c^+_{k_1+p} c_{k_2} \delta_{k_1, k_2-p'} - c^+_{k_2-p'} c_{k_1} \delta_{k_1+p, k_2} \right)$$

$$= \sum_{k_2} \left(c^+_{k_2+p-p'} c_{k_2} - c^+_{k_2-p'} c_{k_2-p} \right)$$

if we replace $k_2 \to k_2 + p$ in the second sum, it seems that the commutator vanishes. We must be careful with ∞ in matrix elements:

$$[\rho^+(p), \rho^+(-p')] = \sum_{k_2} \left(:c^+_{k_2+p-p'} c_{k_2}: - :c^+_{k_2-p'} c_{k_2-p}: \right)$$

$$+ \sum_{k_2} \left(\langle 0| c^+_{k_2+p-p'} c_{k_2} |0\rangle - \langle 0| c_{k_2-p'} c_{k_2-p} |0\rangle \right)$$

the normal ordered contribution vanishes since all matrix elements are finite

As a result we find:

$$[\rho_r^+(p), \rho_{r'}^+(-p')] = \delta_{pp'}\delta_{rr'} \sum_{k_2} \Big(\langle 0| c^+_{r k_2} c_{r k_2} |0\rangle - \langle 0| c^+_{r, k_2-p} c_{r, k_2-p} |0\rangle \Big)$$

Consider now periodic BC $k = \frac{2\pi n}{L}$

if q is occupied $\langle 0| c^+_{r,q} c_{rq} |0\rangle = 1$

$$\boxed{[\rho_r^+(p), \rho_{r'}^+(-p')] = -\delta_{rr'}\delta_{pp'} \frac{s_r p L}{2\pi}}$$

this resembles boson commutation relation $[b, b^+] = 1$ (up to normalization)

in addition

$$\rho_L^+(p>0)|0\rangle = 0$$
$$\rho_R^+(p<0)|0\rangle = 0$$

act as annihilation operators on Dirac see

We define now for $p \neq 0$

$$b_p^+ = \sqrt{\frac{2\pi}{L|p|}} \sum_{s_r} \theta(s_r p)\, \rho_r^+(p)$$

$$b_p = \sqrt{\frac{2\pi}{L|p|}} \sum_{s_r} \theta(s_r p)\, \rho^+(-p)$$

$$[b_p, b_{p'}^+] = \delta_{pp'} \leftarrow \text{canonical relation}$$

Now write everything in terms of bs:

One can show $[b_p, H_{TL}] = v_F \, p \, b_p$, $p>0$

$$\boxed{H_{TL} = \sum_{p\neq 0} v_F |p| \, b_p^\dagger b_p}$$

Unexpected result: in bosonic language free Dirac theory is !quadratic! in b and $b^\dagger$! Since interactions are quadratic in ρ, they are easy to introduce, we get still quadratic theory!

? Can we write <u>elementary fermions</u> in terms of bosonic operators?

use $$[\rho_r^\dagger(p), c_r(x)] = \frac{1}{\sqrt{V}} \sum_{k_1,k_2} e^{ik_2 x} [c^\dagger_{r,k_1+p} c_{rk_1}, c_{rk_2}]$$

$$= -e^{ipx} c_r(x)$$

$$\Downarrow$$

$$\boxed{c_r(x) = e^{\sum_p e^{ipx} \rho_r^\dagger(-p) \left(\frac{2\pi S_r}{pL}\right)}}$$

non-trivial transformation

Subtlety: RHS of the last equation does not change fermion particle number, but c_r should do that. One can introduce an operator O_r (known as the Klein factor) which supresses charge uniformally in space and change total f# by 1. This operator is not important for calculation of space-time dependence of correlation functions, see Giamarchi book for details.

We succeeded in bosonization, but can we go beyond the idealized TL model? Instead of working with b, b^{+}, it is convenient to work with two real angular fields ϕ and θ

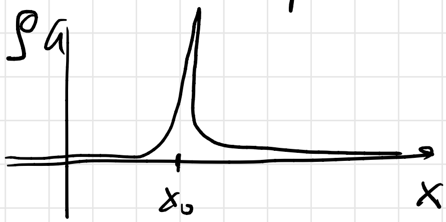
$\phi \rightarrow$ density fluctuations

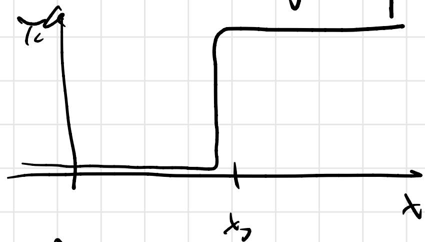
$\theta \rightsquigarrow$ phase fluctuations

$$\cdot\ \rho(x) = \underbrace{\left[\rho_0 - \frac{1}{\pi}\nabla\phi\right]}_{\substack{\text{dominant part}\\ \text{at } q \ll k_F}} \sum_n e^{2in(\pi\rho_0 x - \phi(x))}$$

ρ_0 – constant background

localized particle = π kink of ϕ





$$\bullet\quad \psi^+(x) = \sqrt{\rho(x)}\, e^{-i\theta}$$

θ ← SF phase field

1) we will show below that θ and $\nabla\phi$ are canonically conjugate fields

2) They fluctuate on macroscopic scales $(l > k_F^{-1})$ and thus encode universal long-wavelength physics beyond the TL model

→ Luttinger liquid paradigm

<u>Calculation of commutator of θ and ϕ:</u>

first start from elementary bosonic particle

$$[\psi_B(x), \psi_B^+(x')] = \delta(x-x')$$

$$\psi_B^+ = \sqrt{\rho}\, e^{-i\theta}$$

ρ and θ form the conjugate pair $[\rho(x), \theta(x')] = i\,\delta(x-x')$

Using now that at low-momentum

$$\rho(x) \approx \rho_0 - \frac{1}{\pi}\nabla\phi$$

we get $[\frac{1}{\pi}\nabla\phi(x)\ \theta(x')] = -i\,\delta(x-x')$

for more careful derivation see Giamarchi

We thus found:

1) Canonical momentum of $\phi(x)$ is

$$\Pi(x) = \frac{1}{\pi}\nabla\theta$$

2) $[\phi(x), \theta(x')] = -i\frac{\pi}{2}\,\mathrm{sgn}(x-x')$

Elementary fields in terms of ϕ and θ:

$$\psi_B^+(x) = \sqrt{\rho_0 - \frac{1}{\pi}\nabla\phi}\ \sum_n e^{2in(\pi\rho_0 x - \phi(x))}\ e^{-i\theta(x)}$$

boson

we can also get the representation of fermion by attaching the JW string

$$\psi_F^+(x) = \psi_B^+(x)\, e^{i\pi\int_{-\infty}^{x}\rho(x)} = \psi_B^+(x)\, e^{i\pi\int(\rho_0 - \frac{1}{\pi}\nabla\phi)}$$

$$\psi_F^+(x) = \sqrt{\rho_0 - \frac{1}{\pi}\nabla\phi}\ \sum_n e^{i(2n+1)(\pi\rho_0 x - \phi(x))}\ e^{-i\theta(x)}$$

θ and ϕ can be related to b_q^+ and b_q derived before.

Hamiltonian of the Luttinger liquid

in free theory

- $H_{kin} = \int \frac{\nabla\psi^\dagger \nabla\psi}{2m} \overset{\psi=\sqrt{\rho_0}e^{i\theta}}{=} \frac{\rho_0}{2m}\int (\nabla\theta)^2$

- density-density int $\Rightarrow$ $H_{int} \sim \int (\nabla\phi)^2$

mixed terms $\sim \nabla\phi\nabla\theta$ are prohibited by the inversion symmetry $x \to -x$

Most general Luttinger liquid Hamiltonian

$$H = \frac{1}{2\pi}\int dx \left[uK \underbrace{(\pi\Pi(x))^2}_{(\nabla\theta)^2} + \frac{u}{K}(\nabla\phi)^2 \right]$$

and the corresponding action

$$S = \frac{1}{2\pi K}\int dx\, d\tau \left(\frac{1}{u}(\partial_\tau\phi)^2 + u(\partial_x\phi)^2 \right)$$

relativistic scalar in 1d

u - velocity

K - Luttinger parameter

$K = \begin{cases} > 1 & \text{attraction} \\ < 1 & \text{repulsion} \end{cases}$

This is the most general structure for interacting spinless (either bosonic or fermionic model) in 1d. While in the relativistic TL model u and K can be determined exactly in terms of microscopics, in other int. models these parameters are difficult to compute (can be measured experimentally)

Correlation functions and thermodynamics

ϕ, θ – dimensionless angles

Since the action is quadratic in ϕ, we can extract $\langle\phi\phi\rangle$ from it.

First, introduce $y = u\tau$

$$S = \frac{1}{2\pi K} \int dx\, dy \left((\partial_y \phi)^2 + (\partial_x \phi)^2 \right)$$

in Fourier space $x, y \sim \vec{q}$