free problem, when we set $\omega = 0$ (this is the same as having $V = 0$ everywhere). Thus we are saying

$$\begin{aligned}\langle 0|\exp(-iHT/\hbar)|0\rangle &= \frac{\langle 0|\exp(-iHT/\hbar)|0\rangle}{\langle 0|\exp(-iHT/\hbar)|0\rangle_{\text{free}}}\sqrt{\frac{m}{2\pi i\hbar T}}\\ &= \prod_{n=1}^{\infty}\left[m\left(\frac{n^2\pi^2}{T^2}-\omega^2\right)\right]^{-1/2}\prod_{n=1}^{\infty}\left[m\left(\frac{n^2\pi^2}{T^2}\right)\right]^{1/2}\sqrt{\frac{m}{2\pi i\hbar T}}\\ &= \prod_{n=1}^{\infty}\left[\left(1-\frac{T^2\omega^2}{n^2\pi^2}\right)\right]^{-1/2}\sqrt{\frac{m}{2\pi i\hbar T}}\end{aligned}$$

Now we can use a product formula that is due to L. Euler, who found it in working on the so-called Basel problem (1735):

$$\sin(x) = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right)$$

We thus obtain the final result:

$$\langle 0|\exp(-iHT/\hbar)|0\rangle = \left[\frac{\sin\omega T}{\omega T}\right]^{-1/2}\sqrt{\frac{m}{2\pi i\hbar T}} = \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}$$

In the limit $\omega \to 0$ we see that result reduces to the free problem. In the case, where the $V(x)$ is quadratic, this is an exact result. For other V's this is of course only an approximation.

Many-body physics using the field integral

The view of quantum field theory is that each particle type has an underlying field, which can be disturbed to produce an excitation that we recognize as a particle. If we excite the electromagnetic field we produce a particle called the photon. An example from condensed matter physics, is the concept of lattice vibrations. Since the underlying stuff that is moving are the particles making up the lattice, and since those are described by quantum mechanics, the vibrations themselves are quantized. The corresponding field is called the phonon field and the excitations are called phonons. For the purposes of condensed matter physics these *quasiparticles* are just as real as elementary particles.

We have seen how quantum many-body problem can be formulated in the language of second quantization. For example, the quantum many-body problem of fermions interacting with each other is described by

$$H = \sum_{\boldsymbol{k}}\frac{\boldsymbol{k}^2}{2m}a^{+}_{\boldsymbol{k}\sigma}a_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}}\frac{e^2}{q^2}a^{+}_{\boldsymbol{k}-\boldsymbol{q}\sigma}a^{+}_{\boldsymbol{k}'+\boldsymbol{q}\sigma'}a_{\boldsymbol{k}'\sigma'}a_{\boldsymbol{k}\sigma}.$$

Our task in this section is to find a formulation of this problem in terms of something like a path integral, which is called the field integral. The difference to our previous problem is that we have here a second-quantized Hamiltonian, while before we had a conventional (sometimes called first-quantized) Hamiltonian. Recall how we handled the conventional Hamiltonian: We started with the time evolution operator $\exp(i/\hbar HT)$ that we sliced into small pieces $\exp(i/\hbar H\Delta t)$, which could be expanded in Δt. Then we inserted $|x\rangle$'s and $|p\rangle$'s at the right places that converted the operator H into an ordinary number, since these are the eigenstates of the kinetic and potential energy parts of H.

In order to construct the field integral we will have to find states that turn the second quantized operators H into ordinary numbers. We need a state that is an eigenstate of the second quantized operators. Such a state is called a *coherent state*. These coherent states turn out to be different depending on whether one has bosons or fermions. In the case of fermions one needs to introduce a new kind of mathematical object called a Grassmann variable. We will therefore first tackle the coherent states for bosons. After we have done that, introducing Grassmann variables will be only one additional step that has to be taken in order to arrive at the fermion coherent states.

Coherent states

Let us begin with the simplest case, where our system has only one state and we can fill it up with bosons. We can populate the state by acting with a^+ and we can depopulate the state using a. Then acting with a^+'s on the vacuum of Fock space $|0\rangle$, we obtain consecutively the Fock states with more and more bosons. The state with n-particles is given by

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^+)^n|0\rangle.$$

Let $|\phi\rangle$ now be an eigenstate of a (the operator a^+has no eigenstates. If you haven't seen this, it is another nice exercise to think about). We can represent any Fock space state by a superposition of basis states $|n\rangle$, thus:

$$|\phi\rangle = \sum_n \phi_n|n\rangle \tag{10}$$

We want

$$a|\phi\rangle = \phi|\phi\rangle,$$

where ϕ is the eigenvalue associated with state $|\phi\rangle$. Acting with a on (10) we thus obtain:

$$\phi|\phi\rangle = a|\phi\rangle = \sum_n \phi_n a|n\rangle = \sum_n \phi_n\sqrt{n}|n-1\rangle = \sum_n \phi_{n+1}\sqrt{n+1}|n\rangle$$

Thus by comparing the coefficients on both sides, we obtain

$$\phi\phi_n = \phi_{n+1}\sqrt{n+1}$$
$$\to \phi_{n+1} = \frac{\phi}{\sqrt{n+1}}\phi_n.$$

We can easily solve this recursing by unfolding it step by step:

$$\phi_{n+1} = \frac{\phi}{\sqrt{n+1}}\phi_n = \frac{\phi^2}{\sqrt{n+1}\sqrt{n}}\phi_{n-1} = \cdots = \frac{\phi^{n+1}}{\sqrt{(n+1)!}}\phi_0$$

Thus we have the result that

$$|\phi\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{\sqrt{n!}}|n\rangle = \sum_{n=0}^{\infty} \frac{(\phi a^+)^n}{n!}|0\rangle = \exp(\phi a^+)|0\rangle$$

is an eigenstate of a:

$$a|\phi\rangle = \phi|\phi\rangle.$$

By taking a hermitian conjugate of this equation we obtain another identity:

$$\langle\phi|a^+ = \langle\phi|\phi^*,$$

i.e. the a^+ has a left eigenstate.

The state $|\phi\rangle$ is a superposition of Fock states with different particle numbers. This is a weird concept from the point of view of conventional N particle quantum mechanics, where the number of particles in a system never changes. But it will prove to be an extremely useful concept when dealing with second quantized Hamiltonians that allow for particle number changes.

In the derivation we will need a few identities. Let us discuss these briefly. We will need to know what is the overlap between two different states $|\phi_1\rangle$ and $|\phi_2\rangle$. To compute this we just have to apply the definitions:

$$\langle\phi_2|\phi_1\rangle = \sum_{m=0}^{\infty} \frac{\phi_2^{*m}}{\sqrt{m!}}\langle m| \sum_{n=0}^{\infty} \frac{\phi_1^n}{\sqrt{n!}}|n\rangle = \sum_{n=0}^{\infty} \frac{(\phi_1\phi_2^*)^n}{n!} = \exp(\phi_2^*\phi_1)$$

Thus we see that two coherent states are never orthogonal, they always have a finite inner product.

When we construct the field integral below, we will have to insert resolutions of identity involving coherent states. Thus we need to know how to represent $\mathbb{1}$ by something like $\int d\phi|\phi\rangle\langle\phi|$. The correct identity turns out to be

$$\mathbb{1} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{d\phi_x d\phi_y}{\pi}\ e^{-\phi^*\phi}|\phi\rangle\langle\phi|,$$

where ϕ_x and ϕ_y are the real and imaginary part of ϕ. The integration is thus extended over all the values of ϕ in the complex plane. This completeness relation is a also the reason why ϕ has to be complex. We will now see this in the proof.

To demonstrate this identity we use the definition of $|\phi\rangle$:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{d\phi_x d\phi_y}{\pi}\ e^{-\phi^*\phi}|\phi\rangle\langle\phi| = \sum_{n,m=0}^{\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{d\phi_x d\phi_y}{\pi}\ e^{-\phi^*\phi}\frac{\phi^n}{\sqrt{n!}}|n\rangle\langle m|\frac{\phi^{*m}}{\sqrt{m!}}$$

In order to further evaluate it, we change from the ϕ_x, ϕ_y integration to polar coordinats: $\phi_x = \rho \cos\theta$ and $\phi_y = \rho \sin\theta$:

$$\sum_{n,m=0}^{\infty} \int_0^{\infty} \int_0^{2\pi} \frac{d\rho \rho d\theta}{\pi} \; e^{-\rho^2} \frac{\rho^{n+m} e^{i(n-m)\theta}}{\sqrt{n!}} |n\rangle\langle m| \frac{1}{\sqrt{m!}}$$

The integration over θ kills all terms in the double sum except for the $n = m$ terms:

$$\sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\rho \rho 2\pi}{\pi} \; e^{-\rho^2} \frac{\rho^{2n}}{n!} |n\rangle\langle n|$$

Now we take up the integration over ρ by changing to a variable $u = \rho^2$:

$$\int_0^{\infty} d\rho\rho \; e^{-\rho^2} \rho^{2n} = \frac{1}{2} \int_0^{\infty} du \; e^{-u} u^n = \frac{n!}{2}$$

Thus we obtain

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\phi_x d\phi_y}{\pi} \; e^{-\phi^* \phi} |\phi\rangle\langle\phi| = \sum_{n=0}^{\infty} \frac{n!}{2} \frac{2\pi}{\pi} \frac{1}{n!} |n\rangle\langle n| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1},$$

ie.. we obtain the $\mathbb{1}$ in Fock-space. Note that it was crucial to have complex numbers as the domain of integration. If we had only allowed real ϕ's the phase factor $e^{i(n-m)\theta}$ would not have appeared and we wouldn't have produced the sum over $|n\rangle\langle n|$.

So far we have dealt with the simplest case of bosons occupying a single state created by a. In general a boson can exist in many states and we will have at least one additional quantum number specifying the state. Thus we will have operators a_i^+ with many different values of i, creating bosons in various quantum states. Since boson creation operators commute it is straightforward to generalize the coherent state construction from above to this more general setting. Only the notation gets a little bit more elaborate. The Fock-space is spanned by the occupation number basis states:

$$|n_1 n_2 \ldots\rangle$$

which describes a state occupied by n_1 bosons in state $i = 1$, n_2 bosons in state $i = 2$ and so on. It is formally constructed by acting on the Fock space vacuum with

$$|n_1 n_2 \ldots\rangle = \frac{(a_1^+)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^+)^{n_2}}{\sqrt{n_2!}} \ldots |0\rangle$$

We find that the state defined by

$$|\phi\rangle = \prod_i \exp\left(\phi_i a_i^+\right) |0\rangle = \exp\left(\sum_i \phi_i a_i^+\right) |0\rangle$$

is a coherent state for all the operators a_i. The second equality holds because all the a_i^+ commute with each other. It is clear that $|\phi\rangle$ is a coherent state since a_i commutes with all a_j^+ for $j \neq i$. Thus

$$a_j|\phi\rangle = a_j \prod_i \exp\left(\phi_i a_i^+\right)|0\rangle = \phi_j \prod_i \exp\left(\phi_i a_i^+\right)|0\rangle = \phi_j|\phi\rangle$$

for all j. Thus $|\phi\rangle$ is a coherent state for all annihilation operators.

Returning with this $|\phi\rangle$ to the rules we found above, it is easy to generalize them:

$$\langle\phi'|\phi\rangle = \exp(\sum_i \phi_i'^*\phi_i)$$

The completeness relation becomes

$$\mathbb{1} = \int_{-\infty}^{+\infty} \prod_i \int_{-\infty}^{+\infty} \frac{d\phi_{ix}d\phi_{iy}}{\pi} \exp(-\sum_i \phi_i^*\phi_i)|\phi\rangle\langle\phi| = \int d(\phi^*,\phi)\ \exp(-\sum_i \phi_i^*\phi_i)|\phi\rangle\langle\phi|. \tag{11}$$

In the last equality we have combined all the integrations over the (infinite) product over states i into one symbol $\int d(\phi^*,\phi)$. These are all the results we need to derive the field integral for any body systems.

Field integral for many body systems

Similar to what we have learned for single particle systems, the central object in equilibrium many-body physics is the partition function Z. Once we have a handle on Z we can derive all kinds of equilibrium properties of a many-body system, as for example correlation functions that can be measured in experiments. These correlation functions are similar to the averages that we computed using Wick's theorem. After expressing the partition function as a field integral we can using all those results to compute these averages.

We begin with a second-quantized Hamiltonian. It is enough for us to take a schematic form of the type

$$H = \sum_{ij} T_{ij}a_i^+a_j + \sum_{ijkl} V_{ijkl}a_i^+a_j^+a_ka_l.$$

For instance, the a's could be the annihilation operators for momentum eigenstates. Then the first term would be a kinetic energy and the second term would provide interactions between the particles. The gandcanonical partition function is given by

$$Z = \mathrm{Tr}e^{-\beta(H-\mu N)} = \sum_n \langle n|e^{-\beta(H-\mu N)}|n\rangle,$$

where n are all the Fock space states. The operator N is the particle number operator that counts the number of particles in the system and is given by

$$N = \sum_i a_i^+a_i.$$

The parameter μ is the chemical potential that we can adjust to set the number of particles to the desired value.

As with the single particle problem we perform a change of basis from the states $|n\rangle$ to the coherent states that we have been developing. We do this by inserting the resolution of identity (11):

$$Z = \int d(\phi^*, \phi)\ e^{-\sum_i \phi_i^* \phi_i} \cdot \sum_n \langle n|\phi\rangle\langle\phi|e^{-\beta(H-\mu N)}|n\rangle$$

Now we move $\langle n|\phi\rangle$ to the very right and use $\sum_n |n\rangle\langle n| = 1$ to bring Z into the form:

$$Z = \int d(\phi^*, \phi)\ e^{-\sum_i \phi_i^* \phi_i} \cdot \langle\phi|e^{-\beta(H-\mu N)}|\phi\rangle$$

Notice that $\hbar\beta$ has units of time. In fact the exponential looks like a time evolution operator with *imaginary time*. We therefore proceed as we did with the path integral. We slice the exponential into M factors by dividing the imaginary time $\hbar\beta$ into M parts (we use the symbol M in order to distinguish it from the particle operator N):

$$Z = \int d(\phi^*, \phi)\ e^{-\sum_i \phi_i^* \phi_i} \cdot \langle\phi|e^{-\frac{\beta\hbar}{M}\frac{1}{\hbar}(H-\mu N)} \dots e^{-\frac{\beta\hbar}{M}\frac{1}{\hbar}(H-\mu N)}|\phi\rangle$$

Now we insert the identity (11) between the exponentials. This adds $M-1$ integrations. We label the integrals by $\int d(\phi^{(n)*}, \phi^{(n)})$. Let us look at what happens to an exponential factor when we do this. We will have factors that look like this

$$\langle\phi^{(n+1)}|e^{-\frac{\beta\hbar}{M}\frac{1}{\hbar}(H-\mu N)}|\phi^{(n)}\rangle$$

Now in the limit where M is very large (in the end we let $M \to \infty$), the exponent can be expanded to first order. After that we can let the coherent states $|\phi^{(n)}\rangle$ and $\langle\phi^{(n+1)}|$ act on the Hamiltonian and N operator and re-exponentiate everything. The effect of this is to replace the a^+ and a operators by $\phi^{(n+1)*}$ and $\phi^{(n)}$ respectively. We denote this Hamiltonian by $H(\phi^{(n+1)*}, \phi^{(n)})$ and the N operator by $N(\phi^{(n+1)*}, \phi^{(n)})$. Thus we obtain

$$\begin{aligned}
\langle\phi^{(n+1)}|e^{-\frac{\beta}{M}(H-\mu N)}|\phi^{(n)}\rangle &= e^{-\frac{\beta\hbar}{M}\frac{1}{\hbar}(H(\phi^{(n+1)*},\phi^{n})-\mu N(\phi^{(n+1)*},\phi^{n}))}\langle\phi^{(n+1)}|\phi^{(n)}\rangle \\
&= e^{-\frac{\beta\hbar}{M}\frac{1}{\hbar}(H(\phi^{(n+1)*},\phi^{n})-\mu N(\phi^{(n+1)*},\phi^{n}))} e^{\sum_i \phi_i^{(n+1)*}\phi_i^{(n)}}
\end{aligned}$$

Where we have used the overlap formula to compute $\langle\phi^{(n+1)}|\phi^{(n)}\rangle$. You see that these formulas become somewhat unwieldy, since we are also carrying around a sum over the index i. Let us suppress this sum, we can always put it back later. With this the partition function becomes

$$\begin{aligned}
Z &= \prod_{n=1}^{N} \int d(\phi^{(n)*}, \phi^{(n)})\ e^{-\sum_{n=1}^{M} \phi^{(n)*}\phi^{(n)}} e^{\sum_{n=0}^{M} \phi^{(n+1)*}\phi^{(n)} - \frac{\beta\hbar}{M}\frac{1}{\hbar}(H(\phi^{(n+1)*},\phi^{n})-\mu N(\phi^{(n+1)*},\phi^{n}))} \\
&= \prod_{n=1}^{N} \int d(\phi^{(n)*}, \phi^{(n)})\ e^{\Delta\sum_{n=0}^{M-1}\left[\frac{\left(\phi^{(n+1)*}-\phi^{(n)}\right)}{\Delta}\phi^{(n)} - \frac{1}{\hbar}(H(\phi^{(n+1)*},\phi^{n})-\mu N(\phi^{(n+1)*},\phi^{n}))\right]}
\end{aligned}$$

where we defined $\Delta = \frac{\beta\hbar}{M}$ and by $\phi^{(0)} = M$ we mean ϕ.

Let us now take the continuum limit $M \to$. We obtain

$$Z = \int\limits_{\phi(0)=\phi(\beta\hbar)} D(\phi^*, \phi) e^{-\frac{1}{\hbar} S[\phi^*, \phi]} \tag{12}$$

with

$$S[\phi^*, \phi] = \int\limits_0^{\beta\hbar} d\tau \left[\hbar\phi^* \partial_\tau \phi + H(\phi^*, \phi) - \mu N(\phi^*, \phi)\right]$$

and also we lumped the coherent state integrals into one big functional integral

$$\int D(\phi^*, \phi) = \int d(\phi^{(n)*}, \phi^{(n)}).$$

Since in our derivation we noted that $\phi^{(0)} = \phi^{(M)}$, the integral in (12) has to carried out with the restriction that $\phi(\tau = 0) = \phi(\tau = \beta\hbar)$, i.e. we perform a big functional integral over field configurations with the condition that after the imaginary time $\beta\hbar$ has passed the fields are back to the way they were.

The Field integral for fermions

There is something peculiar that goes on when we try to construct a coherent state for fermionic systems. Let the creation operators for fermions be a_i^+. We know that fermions satsify the Pauli principle and thus have an anti-commutation relation between them:

$$a_i a_j^+ + a_j^+ a_i = \delta_{ij}$$
$$aa = a^+ a^+ = 0$$

Now imagine that $|\psi\rangle$ is a coherent state for fermions, i.e.

$$a_i|\psi\rangle = \psi_i|\psi\rangle$$
$$a_j|\psi\rangle = \psi_i|\psi\rangle.$$

Then let's consider this (for $i \neq j$)

$$a_i a_j|\psi\rangle = \psi_i \psi_j|\psi\rangle$$

But at the cost of a minus sign, we can also reverse the order of a_i and a_j before we let those act on the coherent state:

$$a_i a_j|\psi\rangle = -a_j a_i|\psi\rangle = -\psi_j \psi_i|\psi\rangle$$

Thus

$$\psi_i \psi_j = -\psi_j \psi_i$$

and at this point we could conclude that it is impossible to construct a coherent state for fermions, since ordinary numbers don't anticommute. But it turns out that there is a mathematical system of numbers, called the Grassmann numbers, that have exactly this property. We can use them to construct coherent states and out of that field integrals.

We introduce Grassmann variables by saying that first of all they anticommute:

$$\psi_i \psi_j = -\psi_j \psi_i$$

This implies that the square of a Grassmann variable is 0:

$$\psi^2 = 0$$

Let's look at a function of a Grassmann number $f(\psi)$. We define this by inserting the ψ as the argument in the Taylor expansion of f. It turns out that such functions on Grassman variables have much less complexity than functions defined on ordinary numbers. Since the square of ψ is 0, we are left with only the constant and the first order term in ψ:

$$f(\psi) = f(0) + f'(0)\psi$$

Here $f(0)$ and $f'(0)$ are ordinary numbers and we will say that Grassmann variables commute with ordinary numbers.

We could also generalize this to functions of more variables. To define $f(\psi_1, \ldots, \psi_N)$, a multivariable function, one can expand f by the generalized Taylor expansion. Instead of giving the general formula (which we won't need) let us look at a particular case. Let's say we have a function of two variables $f(x, y)$ and we want the Grassmann version. Then we compute

$$f(\psi_1, \psi_2) = f(0,0) + \left.\frac{\partial f}{\partial x}\right|_{x,y=0} \psi_1 + \left.\frac{\partial f}{\partial y}\right|_{x,y=0} \psi_2 + \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x,y=0} \psi_1 \psi_2$$

All the other terms have higher powers of either ψ_1 or ψ_2 and therefore vanish.

Below we will need to define integrations over Grassmann variables. One of the properties that we have with ordinary integrals over the full domain to obey is this

$$\int dx \; f(x) = \int dx \; f(x + c)$$

(we use this with Gaussian integrals whenever we complete the square). We want the same to hold for Grassmann variables. In fact we can take that as the definition of Grassmann integrals. We know what the most general function of one variable is, so we plug it in and see:

$$\int d\psi \; f(\psi) = \int d\psi \; f(\psi + \psi')$$

Where ψ' is an arbitrary Grassmann number. The left hand side is $f(0)+f'(0)\psi$, while the right hand side is $\int d\psi \; f(0)+f'(0)(\psi+\psi')$. Equating these, we obtain

$$\int d\psi \; f'(0)\psi' = 0$$

Since ψ' is arbitray we have the rule

$$\int d\psi \ 1 = 0.$$

The product of two Grassmann variables commutes with any other Grassmann variables, since there are two minus signs involved. Thus we understand the product of two Grassmann numbers as an ordinary number. Then $\int d\psi \ \psi$ is an ordinary number, which we define to be 1, which is a bit like a normalization of Grassmann numbers:

$$\int d\psi \ \psi = 1$$

How do Grassmann numbers behave when multiplied by a fermion operator? Let's take as an example the coherent state we want $a_j a_i|\psi\rangle = \psi_j\psi_i|\psi\rangle$, so:

$$a_j a_i|\psi\rangle = a_j(\psi_i|\psi\rangle) = -(\psi_i a_j|\psi\rangle) = -(\psi_i\psi_j|\psi\rangle) = (\psi_j\psi_i|\psi\rangle)$$

In the second equality we had to make a anticommute with ψ, otherwise would have obtained the wrong sign. Thus fermion operators and Grassmanns anticommute.

In this way we have constructed a logically consistent mathematical system and we can use it to build up our fermion coherent states. We construct similar to before

$$|\psi\rangle = \exp(\sum_i \psi_i a_i^+)|0\rangle.$$

But for the ket we can't just take the hermitian conjugate, since we don't have the notion of complex conjugation for Grassmann variables. They don't have a real and imaginary part. Thus we do something else instead, we introduce a ket that is independent of the bra and contains a new set of Grassmann variables that have a bar on top, but without the meaning that it's complex conjugation:

$$\langle\psi| = \langle 0|\exp(-\sum_i a_i\bar{\psi}_i) = \langle 0|\exp(\sum_i \bar{\psi}_i a_i)$$

With this we can state the completeness relation

$$\int d(\bar{\psi},\psi) \ \exp(-\sum_i \bar{\psi}_i\psi_i)|\psi\rangle\langle\psi| = \mathbb{1},$$

where $\int d(\bar{\psi},\psi) = \int \prod_i d\bar{\psi}_i d\psi_i$. Thus the only formal change here is the absence of the factor π in the measure. Now one can go through the whole steps similar to the ones we took for the bosonic states to prove the other useful facts about coherent states. These relations are unchanged.

Thus we have enough information now to construct the field integral for the partition function of fermions. Instead of repeating the steps, let us rather look at where the crucial differences are. The first step in our derivation was to insert a resolution of identity into

$$Z = \text{Tr} e^{-\beta(H-\mu N)} = \sum_n \langle n|e^{-\beta(H-\mu N)}|n\rangle,$$

in order to get rid of the states $|n\rangle$. Let's do this:

$$Z = \sum_n \langle n|e^{-\beta(H-\mu N)}|n\rangle = \int d(\bar{\psi},\psi)\ \exp(-\sum_i \bar{\psi}_i\psi_i) \sum_n \langle n|\psi\rangle\langle\psi|e^{-\beta(H-\mu N)}|n\rangle$$

One can prove that moving $\langle n|\psi\rangle$ to the right changes the sign of the Grassmann variables in one of the coherent states. Thus:

$$\langle n|\psi\rangle\langle\psi|m\rangle = \langle -\psi|m\rangle\langle n|\psi\rangle$$

This leads to

$$Z = \int d(\bar{\psi},\psi)\ \exp(-\sum_i \bar{\psi}_i\psi_i)\langle -\psi|e^{-\beta(H-\mu N)}|\psi\rangle$$

Now we can go through the same steps as before to derive the field integral for fermions. We obtain the same result with one very important difference: because we start with $\langle -\psi|e^{-\beta(H-\mu N)}|\psi\rangle$, ψ has to come back to $-\psi$ after at time $\hbar\beta$, i.e. the boundary condition on the field integral has changed:

$$Z = \int\limits_{\psi(0)=-\psi(\beta\hbar)} D(\bar{\psi},\psi)e^{-\frac{1}{\hbar}S[\psi^*,\psi]} \tag{13}$$

with

$$S[\psi^*,\psi] = \int\limits_0^{\beta\hbar} d\tau\ [\hbar\psi^*\partial_\tau\psi + H(\psi^*,\psi) - \mu N(\psi^*,\psi]$$

Thus in this field integral approach to physics the only difference between bosons and fermions is a minus sign in the boundary conditions of the field integral. Bosons are periodic in imaginary time, while fermions are antiperiodic.