Quantum fields

Many-body problem gave birth to quantum field theory

$\varphi(x)$ – classical field

$\big\downarrow$ via second quantization promoted to operator

$\hat{\varphi}(x) \to$ destroys a particle at position x, fluctuates

It was Einstein who in 1905 and 1907 introduced quanta of energy $E = \hbar\omega$ for el-mag and crystal matter.

First quantization recap:

$$p \to \hat{p} = -i\hbar\partial_x \Rightarrow [\hat{x}, \hat{p}] = i\hbar$$

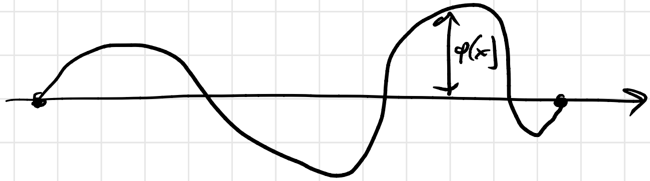
$$\Downarrow$$

Heisenberg uncertainty rel. $\Delta x\, \Delta p \geq \frac{\hbar}{2}$

We want however treat a many-body problem and usually Sch. eq is useless.

Instead, we are interested in 1) excitations above the ground state, 2) correlations and response to ext. probes 3) collective behavior

Quantization of a classical string:



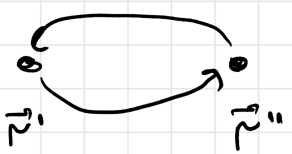
$$H = \int dx \left[\frac{T}{2} (\nabla_x \hat{\varphi})^2 + \frac{1}{2\rho} \hat{\pi}^2(x) \right]$$

$$\boxed{[\hat{\varphi}(x), \hat{\pi}(y)] = i\hbar\, \delta(x-y)}$$ x, y are just labels

works well for quantizing sound vibrations, el-mag. fields

? What is a quantum field theory of particles, e.g. electrons, He atoms, etc?

Quantum statistics recap: In QM there are two fundamentally different types of particles - fermions and bosons



$$\psi(\vec{r}\,', \vec{r}\,'') = e^{i\theta} \psi(\vec{r}\,'', \vec{r}\,')$$

$$e^{i\theta} = \begin{cases} +1 & \text{bosons} \\ -1 & \text{fermions} \end{cases}$$ not 2d!

Double exchange $e^{2i\theta} = +1$

In 1927 Jordan and Klein proposed

$$\psi(x) \rightarrow \hat{\psi}(x)$$ second quantization

$\hat{\psi}$ and $i\hat{\psi}^{+}$ is a pair of canonically conjugate variables $\mathcal{L} = \frac{i}{2}\psi^{+}\overleftarrow{\partial_t}\psi$

But how to distinguish bosons and fermions?

naive $[,]$ $\qquad [\hat{\psi}(x),\hat{\psi}(y)]=0$

$$\boxed{\begin{array}{ll} \text{For bosons} & [\hat{\psi}(x),\hat{\psi}^{+}(y)] = \delta(x-y) \\ \text{For fermions} & \{\hat{\psi}(x),\hat{\psi}^{+}(y)\} = \delta(x-y) \end{array}}$$

anticommutator $\qquad \{\hat{\psi}(x),\hat{\psi}(y)\}=0$

To understand why it is so, we make connection to many-body wave-function.

First, introduce vacuum state $|0\rangle$

$$\hat{\psi}(x)\,|0\rangle = 0$$ a state with no particles

Now we can create particles with $\hat{\psi}^{+}$

$$|x_1, x_2, \dots x_N\rangle = \hat{\psi}^{+}(x_N)\dots\hat{\psi}^{+}(x_1)|0\rangle$$

and corresponding bra

$$\langle x_1, \dots x_N| = \langle 0|\,\hat{\psi}(x_1)\dots\hat{\psi}(x_N)$$

The N-body wavefunction for a state $|N\rangle$

$$\psi(x_1, \dots x_N) = \langle 0| \hat{\psi}(x_1) \dots \hat{\psi}(x_N) |N\rangle$$

<u>Wavefunction encodes matrix elements of quantum fields</u>

Now we can understand why fermion operators anticommute

$$\psi(x_1, x_2) = - \psi(x_2, x_1)$$

$$\Updownarrow$$

$$\hat{\psi}(x_1)\hat{\psi}(x_2) = -\hat{\psi}(x_2)\hat{\psi}(x_1)$$

So <u>quantum statistics dictates (anti)commutation relations of fields</u>

Out of elementary field operators we can construct more complicated operators, e.g. density operator

$$\hat{\rho}(x) = \hat{\psi}^{+}(x)\hat{\psi}(x)$$

whose expectation value in a state $|N\rangle$

$$\rho(x) = \langle N| \hat{\rho}(x) |N\rangle$$

Unlike classical fields, quantum fields fluctuate and so the canonical pair $\hat{\varphi}, i\hat{\varphi}^{\dagger}$ cannot have a sharp expectation value. Lets introduce

$$\hat{\varphi}(x) = \sqrt{\hat{\rho}(x)}\, e^{i\hat{\theta}(x)}$$

for bosons com. relations of $\hat{\varphi}, \hat{\varphi}^{\dagger}$

$$\boxed{[\hat{\rho}(x), \hat{\theta}(y)] = i\delta(x-y)}$$

which implies $\Delta N \Delta \theta \gtrsim 1$, we will see later that in a superfluid ΔN is large $\Rightarrow \Delta\theta \to 0$, i.e. the phase field $\hat{\theta}$ acquires macroscopic coherence. Similar to laser, in superfluids we can observe interference!

Now we will demonstrate that field equations of quantum fields imply many-body Sch. equation.

$$\hat{H} = \int_x \hat{\psi}^\dagger \left(-\frac{\hbar^2 \nabla^2}{2m} + U(x)\right)\hat{\psi} + \frac{1}{2}\int_{x,x'} \underbrace{V(x-x')}_{\text{int. potential}} :\hat{\rho}(x)\,\hat{\rho}(y):$$

where in normally ordered operator $:\mathcal{O}:$ all destruction operators are on the right

$$:\hat{\rho}(x)\hat{\rho}(y): \;=\; \underset{\text{depends on statistics}}{\mp}\; \psi^\dagger(x)\,\psi^\dagger(y)\,\psi(x)\,\psi(y) = \psi^\dagger(y)\,\psi^\dagger(x)\,\psi(x)\,\psi(y)$$

Using Heisenberg equation of motion

$$i\hbar\,\partial_t \hat{\psi} = [\hat{\psi}, H]$$

$$\Downarrow$$

$$i\hbar\,\partial_t \hat{\psi}(x) = \left[-\frac{\hbar^2\nabla^2}{2m} + U(x)\right]\hat{\psi}(x) + \int dx'\, V(x'-x)\,\hat{\rho}(x')\,\hat{\psi}(x)$$

Now lets take $i\hbar\partial_t$ of many-body wavefunction

$$i\hbar\,\partial_t \Psi(x_1,\dots x_N) = i\hbar\sum_{j=1}^{N} \langle 0|\hat{\psi}(x_1)\dots\partial_t\hat{\psi}(x_j)\dots\hat{\psi}(x_N)|\Psi\rangle$$

$$= \sum_{j=1}^{N}\left[-\frac{\hbar^2\nabla_j^2}{2m} + U(x_j)\right]\Psi +$$

$$+ \sum_j \int dx'\, V(x'-x_j)\,\langle 0|\hat{\psi}(x_1)\dots\hat{\rho}(x')\hat{\psi}(x_j)\dots\hat{\psi}(x_N)|\Psi\rangle$$

Now we commute to the density to the left

$$\langle 0|\hat{\varphi}(x_1)\ldots\hat{\rho}(x')\hat{\varphi}(x_j)\ldots\hat{\varphi}(x_N)|\Psi\rangle =$$

$$= \sum_{\ell<j}\delta(x'-x_\ell)\langle 0|\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_N)|\Psi\rangle$$

As a result, we get many-body Sch. eq

$$\boxed{i\hbar\partial_t\Psi = \left(\sum_{j=1}^{N}\mathcal{H}_j^{(0)} + \sum_{\ell<j}V_{\ell j}\right)\Psi}$$

The final result does not depend on quantum statistics

Grand canonical ensemble from 2nd quantization

Grand canonical ensemble — a system is in contact with heat bath which it can exchange energy and particles.

Probability of being in a state λ of energy E_λ and particle number N_λ

$$\boxed{p_\lambda = \frac{1}{Z}e^{-\beta(E_\lambda-\mu N_\lambda)}}, \quad \beta = 1/k_BT$$

$$Z = \sum_\lambda e^{-\beta(E_\lambda-\mu N_\lambda)} \rightsquigarrow \text{normalization constant}$$

partition function