C. Lehmann Book

Green's function

In translation and time-invariant system (Heisenberg representation)

$$G(\vec{x}-\vec{x}',\ t-t') = -i \langle GS | T\, \hat{\psi}(\vec{x},t)\, \hat{\psi}^{+}(\vec{x}',t') | GS\rangle$$

$|GS\rangle$ many-body ground state

it is a quantum-mechanical amplitude of propagation of single particle on top of many-body ground state from spacetime point $(\vec{x}', t')$ to spacetime point $(\vec{x}, t)$

T denotes time-ordering

$$T\,\hat{\psi}(t)\,\hat{\psi}^{+}(t') = \begin{cases} \hat{\psi}(t)\,\hat{\psi}^{+}(t') & (t>t') \\ \pm\,\hat{\psi}^{+}(t')\,\hat{\psi}(t) & (t<t') \end{cases}$$

Important information that we can extract from G: • density

$$\langle \hat{\rho}(x) \rangle = \langle \hat{\psi}^{+} \hat{\psi} \rangle = -\langle \varphi | T\, \hat{\psi}(x,0^{-})\, \hat{\psi}^{+}(x,0) | \varphi \rangle$$
$$= -i\, G(0,0^{-})$$

similar → • kinetic energy density

$$\langle \hat{T}(x) \rangle = -\frac{\hbar^2}{2m} \langle \hat{\psi}^{+}_{x}\, \nabla^2_{x}\, \hat{\psi} \rangle = i\,\frac{\hbar \nabla^2}{2m}\, G(x,0^{-})\Big|_{x=0}$$

- One-particle spectral function

$$A(k,\omega) = \frac{1}{\pi} \operatorname{Im} G(k, \omega - i\delta)$$

$\rightsquigarrow$ allows to understand excitation spectra of many-body system

Here

$$G(k,\omega) = \int dt\, dx\, G(x,t) e^{-i(kx-\omega t)}$$

Green's function of free fermions

We will work in grand-canonical ensemble:

$$H = H_0 - \mu N = \sum \epsilon_k c_k^\dagger c_k \,; \quad \varepsilon_k = \frac{k^2}{2m} - \mu$$

The ground state is filled FS:

$$|GS\rangle = \prod_{|k|<k_F} c_k^\dagger |0\rangle$$

In Heisenberg picture $\quad \mathcal{O}(t) = e^{iHt} \mathcal{O} e^{-iHt}$

$$c_k^\dagger(t) = e^{i\epsilon_k t} c_k^\dagger$$

$$c_k(t) = e^{-i\epsilon_k t} c_k$$

if $t>t'$ – we can only add a fermion above FS

$$\langle GS|\, c_k(t)\, c^+_{k'}(t')\,|GS\rangle = \delta_{kk'} e^{-i\epsilon_k(t-t')} \otimes$$

$$\underbrace{\langle GS|\, c_k\, c^+_{k'}\,|GS\rangle}_{1-n_k}$$

if $t<t'$ – we can only destroy a fermion inside of a fermi see

$$\langle GS|\, c^+_{k'}(t')\, c_k(t)\,|GS\rangle = \delta_{kk'}\, n_k\, e^{-i\epsilon_k(t-t')}$$

putting it together

$$G(k,t) = -i\Big[\underbrace{(1-n_k)\,\theta(t)}_{\text{particle contribution}} - \underbrace{n_k\,\theta(-t)}_{\text{hole contribution}}\Big] e^{-\epsilon_k t}$$

It is useful to Fourier $t \to \omega$:

$$\boxed{G(k,\omega) =} \int dt\, e^{i\omega t}\, G(k,t)$$

$$= -i \int dt\, e^{i(\omega-\epsilon_k)t}\, \underbrace{e^{-|t|\delta}}_{\text{convergence factor}} \left[\theta_{k-k_F}\,\theta(t) - \theta_{k_F-k}\,\theta(-t)\right]$$

$$= -i\left[\frac{\theta_{k-k_F}}{\delta - i(\omega-\epsilon_k)} - \frac{\theta_{k_F-k}}{\delta + i(\omega-\epsilon_k)}\right] = \boxed{\frac{1}{\omega-\epsilon_k+i\delta_k}},$$

where $\delta_k = \delta\, \operatorname{sgn}(k-k_F)$

But how to compute the Green's function in an interacting problem

$$H_{int} = \int dx\, dx'\, c^\dagger(x) c(x) \underbrace{V(x-x')}_{\text{interaction potential}} c^\dagger(x') c(x')$$

1) Turn on H_{int} adiabatically $\rightarrow$ Gell-Mann-Low result

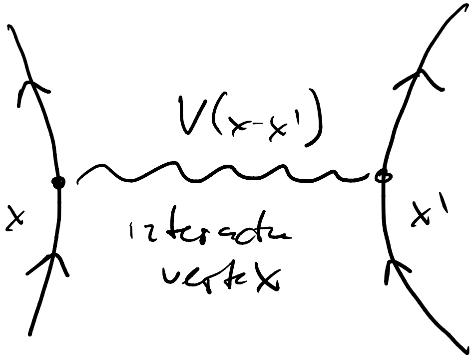
$$\langle GS | T \psi(t)\, \psi^\dagger(t') | GS \rangle = \frac{\langle GS_0 | T\, \hat{S}\, \psi_0(t)\, \psi_0^\dagger(t') | GS_0 \rangle}{\langle GS_0 | \hat{S} | GS_0 \rangle}$$

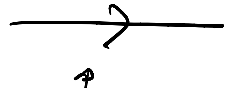
where $\underbrace{\hat{S}}_{\text{S-matrix}} = T \exp\left(-i \int_{-\infty}^{+\infty} H_{int}(t')\, dt' \right)$

LHS $\rightarrow$ Greens function of interacting problem

RHS $\rightarrow$ the ratio of complicated expectation values in the non-interacting problem.

Let's do perturbative calculation in H_{int} and use Feynman diag.

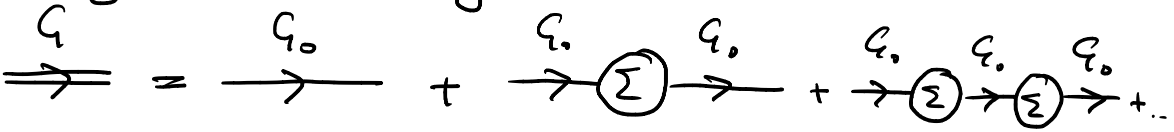




$= G_0$

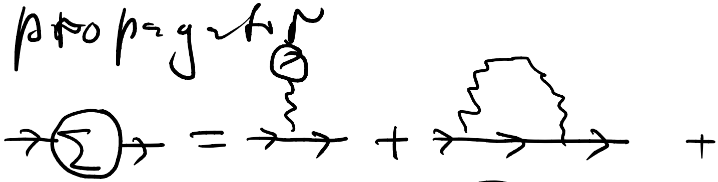
free propagator

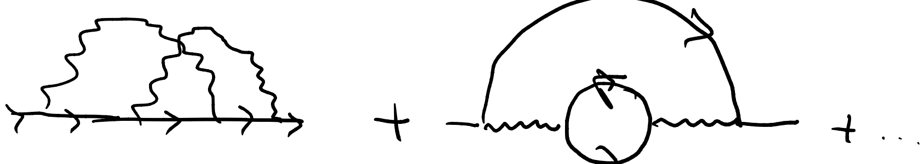
The full Green's function is obtained from the sum of Feynman diagrams



where we introduced the self-energy - the sum of all (1PI) scattering processes that cannot be fully separated by cutting a single fermion propagator

$\Sigma(k,\omega) =$





external legs must be amputated

The self-energy is complicated, but if we know it we can resum the series

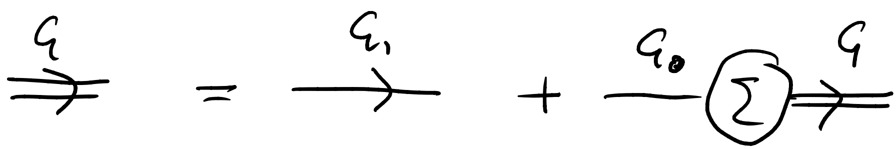
$$G = G^0 + G^0 \Sigma G^0 + G^0 \Sigma G^0 \Sigma G^0 + \dots$$

$$= \frac{G_0}{1 - \Sigma G_0} = \frac{1}{G_0^{-1} - \Sigma}$$

or substituting G_0 explicitly

$$G(k,\omega) = \frac{1}{\omega - \epsilon_k - \Sigma(k,\omega)}$$

this can also be written as



$$G = G_0 + G_0 \Sigma G$$

which is known as the <u>Dyson equation</u>

Physical meaning of the self-energy:

Σ describes <u>the cloud of particle-hole excitations which accompanies the fermion</u>

$$\Sigma(k, \omega - i\delta) = \underset{\text{real part}}{\Sigma'(k,\omega)} + i\,\underset{\text{imaginary part}}{\Gamma(k,\omega)}$$

$$A(k,\omega) = \frac{1}{\pi}\,\mathrm{Im}\, G(k, \omega - i\delta)$$

$$= \frac{1}{\pi}\,\frac{\Gamma}{|\omega - \epsilon_k - \Sigma'|^2 + \Gamma^2}$$

Lorentzian of <u>width Γ</u> concentrated at renormalized energy <u>$\varepsilon_k^* = \epsilon_k + \Gamma(k, \epsilon_k)$</u>