Fermi liquid theory

One of the first low-energy effective theory of quantum matter was developed by Landau in 1956.
Applicable to neutral systems of fermions (e.g. ^3He, cold atoms) and charged states (metals).
Does not assume weak interaction, e.g. usually in metals Coulomb interaction ~ kinetic energy.
Landau Fermi liquid theory is a theory of fluctuating Fermi surface → "string theory" of condensed matter physics.

Basic idea: use adiabatic continuity

free Fermi gas ⟶ interacting Fermi system

Pines / Nozieres

Fermi liquid is qualitatively similar to a free fermion system — there is a Fermi sea and excitations are quasiparticles with a sharp value of charge and spin. Importantly, the energy $E[n_p]$ __is not__ a linear functional of the occupation distribution n_p and quasiparticles acquire __finite decay width__, which however goes to zero as one approaches the Fermi surface

$$\frac{1}{\tau} \ll |\epsilon - \epsilon_F|$$

as we show later this relies on phase space arguments near Fermi surface. In addition, the mass, magnetic moment of excitations are renormalized.

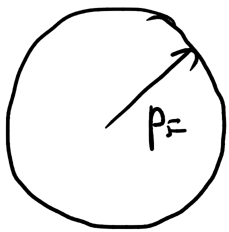
We start from free Fermi gas

$$n_p = \frac{1}{1 + e^{(\epsilon_p - \mu)/k_B T}}$$

and consider quantum degenerate regime

$$\mu > 0 \qquad k_B T \ll \mu$$

for simplicity consider isotropic system (^3He)

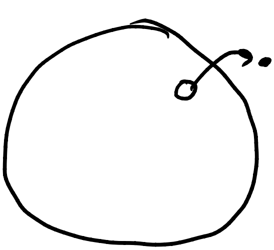


spherical Fermi surface with p_F fixed by density n

GS = <u>all states</u> with $|\vec{p}| < p_F$ are occupied

$$E_0 = \int_{\vec{p}} n_p \frac{p^2}{2m}$$

Excitations are <u>particle-hole pairs</u>



$$\Delta E_{ph} = \int_{\vec{p}} \delta n_p \frac{p^2}{2m} > 0$$

Adiabatic principle — start from free Fermi gas and slowly switch on interaction. We assume that the GS smoothly evolves during this process. This is not always true (e.g. superconductivity), but often works (normal systems)

In isotropic case the Fermi surface does not change under this process since its volume is fixed by the particle number and interactions cannot change the volume (Luttinger theorem).

Under adiabatic evolution

particle exc $\rightsquigarrow$ quasiparticles
hole exc. $\rightarrow$ quasiholes

that carry same momentum, charge and spin as original free fermions

Landau ~~energy~~ excitation functional of the Fermi liquid theory: Taylor expansion

$$\Delta E[\delta n_p] = \int_{\vec{p}} \epsilon_p^\sigma \, \delta n_p^\sigma + \frac{1}{2} \int_{\vec{p},\vec{p}'} f_{\vec{p}\vec{p}'}^{\sigma\sigma'} \, \delta n_{\vec{p}}^\sigma \, \delta n_{\vec{p}'}^{\sigma'}$$

individual q-p and q-h

interaction between qp and qh

where $\epsilon_p \equiv \dfrac{\delta E}{\delta n_p}$ → single qp energy

of the group velocity $\vec{v}_p = \dfrac{\partial \epsilon_p}{\partial \vec{p}} = \dfrac{\vec{p}}{m_*}$

isotropy

$m_* = m$, for example $m_* = (3\text{-}6)\, m_{3He}$

depends on pressure

and mean-field interactions are encoded in Landau parameters $f_{\vec{p},\vec{p}'}^{\sigma\sigma'} = \dfrac{\delta^2 E}{\delta n_{\vec{p}}^\sigma \, \delta n_{\vec{p}'}^{\sigma'}}$

It is convenient to introduce Hamiltonian of quasiparticles

$$\tilde{\varepsilon}_{\vec{p},\sigma} \equiv \varepsilon_{\vec{p},\sigma} + \int_{\vec{p}'} f_{\vec{p}\vec{p}'}^{\sigma\sigma'} \, \delta n_{\vec{p}',\sigma'}$$

In general $\delta n_{\bar p,\sigma}$ can depend on positn $\bar r$ and its time evolution is govern by Boltzmann equation

LHS - continuity equation in phase space

$$\partial_t n_{\bar p,\sigma} + \partial_{\bar p}\tilde\varepsilon_{\bar p\sigma}\cdot\partial_{\bar r} n_{\bar p,\sigma} - \partial_{\bar r}\tilde\varepsilon_{\bar p,\sigma}\cdot\partial_{\bar p} n_{\bar p,\sigma} = I(n_{\bar p,\sigma})$$

collision term

Now we linearize it around equilibrium

$$n_{\bar p\sigma}(\bar r,t) = \underset{\text{equilibrium}}{n^{(0)}_{\bar p,\sigma}} + \delta n_{\bar p,\sigma}(\bar r,t)$$

$$\partial_t\delta n_{p\sigma} + v_{\bar p}\cdot\partial_{\bar r}\delta n_{p,\sigma} - \int_{p'\sigma'} f^{\sigma\sigma'}_{pp'}\,\partial_{\bar r}\delta n_{p'\sigma'}\cdot\partial_p n^{(0)}_{p,\sigma} = I[\delta n_{p,\sigma}]$$

$T=0$:

$$\frac{\partial n^{(0)}_{p,\sigma}}{\partial \epsilon_p}\frac{\partial \epsilon_p}{\partial \bar p} = -\delta(\epsilon_p-\epsilon_F)\,\vec v_{\bar p}$$

since collision term should vanish in equilibrium state linearized Boltzmann equation:

$$\boxed{\partial_t \delta n + \vec{v}\,\partial_{\vec{r}}\delta n + \delta(\epsilon_p - \epsilon_F)\,\vec{v}_{\vec{p}} \cdot \int_{p',\sigma'} f^{\sigma\sigma'}_{pp'}\,\partial_{\vec{r}}\,\delta n_{\vec{p}'\sigma'} = I[\delta n]}$$

Since qp are only well-defined only close to the Fermi surface, it is good that $n^0_{\vec{p},\sigma}$ disappeared from the linearized kinetic equation

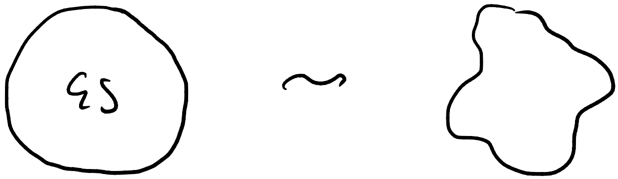
We consider now the collisionless limit

$\omega\tau \gg 1$ $\qquad$ $\tau \to$ collision time from the collision term

$\omega \to$ frequency of interest

in this limit we can neglect the collision term $I[\delta n]$ and the evolution of $\delta n_{\vec{p},\sigma}$ is determined by continuity of probability in phase space.

Consider collective oscillations of the Fermi sea



introduce sharply localized excitations at the Fermi surface

$$\delta n_{\vec{p},\sigma} = \delta(\epsilon_p - \epsilon_F)\, v_F \, u_{\hat{p},\sigma}$$

$$\hat{p} = \frac{\vec{p}}{|\vec{p}|}$$

$$\int f^{\sigma\sigma'}_{\vec{p}\vec{p}'} \, \partial_r \, \delta n_{\vec{p}'\sigma'} = N(\epsilon_F) \int d\epsilon \; \delta(\epsilon - \epsilon_F) \times$$

$$\int \frac{d\Omega'}{4\pi} \sum_{l,\sigma'} \frac{F_l^{\sigma\sigma'}}{2N(\epsilon_F)} P_l(\hat{p}\cdot\hat{p}') \, v_F \, \partial_r u_{\hat{p}',\sigma'}$$

here we expanded the Landau parameters at Fermi momentum in Legendre polynomials since the depend only on $\hat{p}\cdot\hat{p}'$

For spin-invariant interactions

$$F_l^{\sigma\sigma'} = F_l^{(s)} + \sigma\cdot\sigma' \, F_l^{(a)}$$

Consider a plane-wave ansatz:

$$u_{\hat{p},\sigma} \sim e^{i(\vec{q}\cdot\vec{r}-\omega t)}$$

$$\Big(\underbrace{v_F\hat{p}}_{\vec{v}_F}\cdot\vec{q}-\omega\Big)u_{\hat{p},\sigma}+v_F\hat{p}\cdot\vec{q}\int\frac{d\Omega'}{8\pi}\sum f_e^{\sigma\sigma'}P_e(\hat{p}\cdot\hat{p}')u_{\hat{p}'\sigma'} = 0$$

We introduce now:

$$\lambda = \omega/qv_F \qquad \cos\theta = \hat{p}\cdot\hat{q}$$

$$u_{\hat{p}}^{(s)} = \frac{1}{2}\left(u_{\hat{p}\uparrow}+u_{\hat{p}\downarrow}\right) \qquad u_{\hat{p}}^{(a)} = \frac{1}{2}\left(u_{\hat{p}\uparrow}-u_{\hat{p}\downarrow}\right)$$

$$\left(\cos\theta-\lambda\right)u_{\hat{p}}^{(s,a)}+\frac{\cos\theta}{8\pi}\int d\Omega'\sum_e f_e^{(s,a)}P_e(\hat{p}\cdot\hat{p}')u_{\hat{p}'}^{(s,a)}=0$$

will be viewed as eigenvalue equation

expand $u_{\hat{p}}^{(s,a)}$ in spherical harmonics $Y_{l,m}(\theta,\varphi)$

$\rightsquigarrow$ complicated equation for a general set of Landau parameters we will now consider a simple model, where only $F^{(s)}_{\ell=0} \neq 0$

change of chemical potential

Zero sound calculation

$$(\cos\theta - \lambda)\, u^{(s)}_{\hat{p}} + F^{(s)}_0 \frac{\cos\theta}{2} \int \frac{d\Omega'}{4\pi}\, u^{(s)}_{\hat{p}'} = 0$$

$$u^{(s)}_{\hat{p}} = C \frac{\cos\theta}{\cos\theta - \lambda}$$

to determine λ, we substitute the solution back in to the B. equation

$$1 + \frac{F^{(s)}_0}{2} \int \frac{d\Omega'}{4\pi} \frac{\cos\theta'}{\cos\theta' - \lambda} = 0$$

since

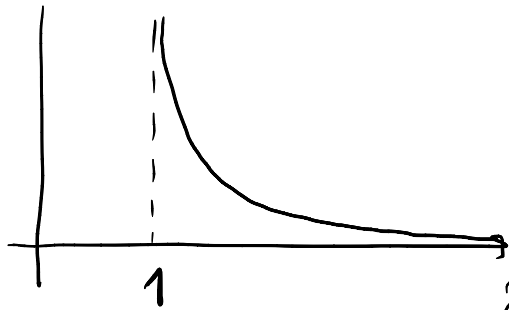
$$\int \frac{d\Omega'}{4\pi} \frac{\cos\theta'}{\cos\theta' - \lambda} = \frac{1}{2} \int_{-1}^{1} \frac{x}{x - \lambda}\, dx =$$

$$= 1 + \frac{\lambda}{2} \ln\left(\frac{\lambda - 1}{\lambda + 1}\right)$$

we get the equation for λ:

$$\boxed{\frac{1}{F_0^{(s)}} = \frac{\lambda}{2} \ln\left(\frac{\lambda+1}{\lambda-1}\right) - 1}$$

RHS



$\lambda = \dfrac{\omega}{q v_F}$

The equation has three qualitatively different regimes:

① $F_0^{(s)} > 0$ repulsive interactions

$\lambda(F_s)$ – single-valued real function

$\lambda > 1$ $\rightsquigarrow$ <u>sharp zero sound mode</u>

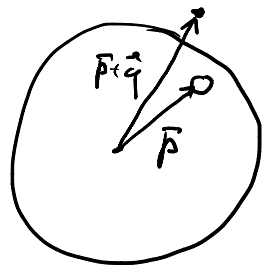
② Weak attraction $0 > F_0^{(s)} > -1$

λ is complex, but <u>$\mathrm{Im}\,\lambda < 0$</u>

<u>strongly damped</u> collective mode

Landau damping — collective mode is damped because it can excite particle-hole excitations.

Energy of a particle-hole pair with momentum $\vec{q}$:

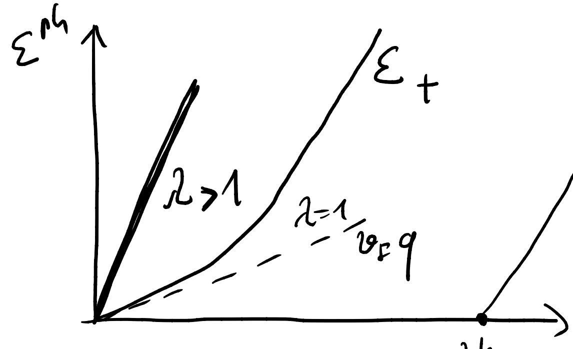


$$\varepsilon^{ph}_{\vec{q}} = \varepsilon_{\vec{p}+\vec{q}} - \varepsilon_{\vec{p}}$$

$|\vec{p}| < p_F \qquad |\vec{p}+\vec{q}| > p_F$

For a given q the p-h energy is largest if $\vec{p}$ and $\vec{q}$ are colinear

$$\varepsilon_+ = \frac{(p+q)^2}{2m} - \frac{p^2}{2m} = \frac{q^2}{2m} + \underbrace{\frac{p}{m}}_{\sim v_F} \cdot q$$



$\lambda > 1$ — zero sound is outside the particle-hole continuum

$\lambda < 1$ — inside the continuum it can decay into p-h pairs

③ $F_0^{(s)} < -1 \Rightarrow \operatorname{Im} \lambda > 0$

the mode does not decay, but exponentially grow! This is a manifestation of <u>Pomeranchuk instability</u> — Fermi surface is unstable to deformations and Fermi liquid paradigm breaks down

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<u>Zero sound</u> calculation is valid in the collisionless limit $\omega\tau \gg 1$

In the opposite (hydrodynamic) regime $\omega\tau \ll 1$ one finds the <u>first sound</u>

$$\lambda = \frac{\omega}{q v_F} = \sqrt{\frac{1 + F_0^{(s)}}{3}}$$

As ω grows the hydro first sound <u>crosses over</u> at $\omega \sim \tau^{-1}$ to the collisionless zero sound, In experiments with ^3He (PRL 17, 74 (1966))

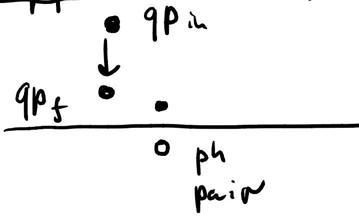
$\tau^{-1} \sim T^2$ as $T \searrow$ $\omega\tau \nearrow$

and we have the first sound at high T and zero sound at low T.

Life-time of Landau quasiparticles

(Coleman book)

Landau idea relies on the adiabatic approximation, but does it work?



quasiparticle can decay into qp + ph pair

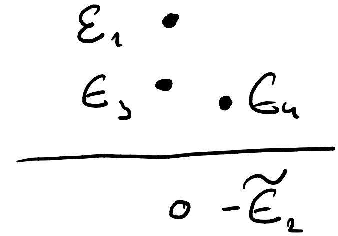
What is the lifetime of qp?

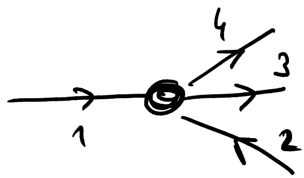
If $\tau \sim \epsilon_{qp}^{-1} \Rightarrow$ cannot use adiabatic approximation

It turns out that due to the Pauli principle $\tau \sim \dfrac{\epsilon_F}{\epsilon_{qp}^2}$. As a result

$\epsilon_{qp}\tau \to \infty$ and adiabaticity works. Qp is sharp close to the Fermi surface, this explains the success of the Landau idea

Calculation of the decay rate:





$$\Gamma_{1\to\bar{2}34}(\epsilon_1) \overset{\text{Fermi Golden rule}}{\sim} \frac{2\pi}{\hbar} \sum_{\epsilon_1 > \tilde{\epsilon}_2\, \epsilon_3 \epsilon_4 > 0} \underbrace{|a(1;\bar{2}34)|^2}_{\text{scattering amplitude}} \delta\left(\epsilon_1 - [\tilde{\epsilon}_2 + \epsilon_3 + \epsilon_4]\right)$$

Since all qp and qh are close to FS

$$|a(1;\bar{2}34)|^2 \xrightarrow[\text{on FS}]{\text{average}} \langle |a_{\mathrm{I}\to\mathrm{III}}|^2 \rangle$$

due to a slow dependence of the amplitude on energies close to Fermi surface

$$\Gamma_{1\to\bar{2}34}(\epsilon_1) = \frac{2\pi}{\hbar} \langle |a_{\mathrm{I}-\mathrm{III}}|^2 \rangle \underbrace{\int d\tilde{\epsilon}_2\, d\epsilon_3\, d\epsilon_4\, \delta\left(\epsilon_1 - \{\tilde{\epsilon}_2 + \epsilon_3 + \epsilon_4\}\right)}_{\sim\, \epsilon_1^2}$$

by dimensional arguments

$$\boxed{\Gamma(\epsilon) \sim \frac{1}{\hbar} \frac{\epsilon^2}{\epsilon_F}}$$ and qp are long-lived as $\epsilon \to 0$

for a decay qp $\to$ n p-h pairs + qp

$$\Gamma_n \sim \frac{1}{\hbar} \frac{\epsilon^{2n}}{\epsilon_F^{2n-1}} \ll \Gamma_{1-\bar{2}34}$$

a detailed calculation was done by Abrikoson and Khalatnikov 1957. At finite temperature they found

$$\Gamma \sim \# \epsilon^2 + \# (k_B T)^2$$

at large temperature $\Gamma \sim T^2$ which implies that resistivity of Fermi liquid $\rho \sim T^2$.

Coulomb interactions (Levitov)

So far we considered neutral Fermi liquids (^3He), in metals we must include Coulomb interactions.

The Boltzmann equation in the collisionless regime is:

$$\partial_t n + \vec{v} \cdot \partial_{\vec{r}} n + e \vec{E} \cdot \partial_{\vec{p}} n = 0$$

where $\vec{E} = -\vec{\nabla} \varphi$ with the scalar potential

$$\varphi = e \int \left[\int n(\vec{p}, \vec{r}') \frac{d^3p}{(2\pi)^3} - n_0 \right] \frac{1}{|\vec{r} - \vec{r}'|} d^3 r'$$

positive ion background

as a result the Landau parameter $F^{(s)}_{\ell\to 0} = \frac{4\pi e^2 N}{q^2}$, where N is total density of states. Zero sound dispersion can now be extracted from $\lambda = \omega / v_F q$

$$\frac{\lambda}{2} \ln\left(\frac{\lambda+1}{\lambda-1}\right) - 1 = \frac{q^2}{4\pi e^2 \partial}$$

Now as $q \to 0$ $\omega \to \omega_0 = \sqrt{\frac{4\pi n e^2}{m}}$

zero sound becomes the gapped plasma mode.

Non-Fermi liquids

Quantum fermionic many-body systems with a Fermi surface, but no well-defined quasiparticles.

We know from before decay into p-h pairs does not destabilize FL.

We will see later that FL has superconducting instability under infinitesimal attractive interaction but there we loose Fermi surface (fermions acquire SC gap), so a superconductor <u>is not</u> a non-Fermi liquid.

How to create a non-Fermi liquid?

! we need a gapless (damped) boson that couples to fermion quasiparticles

*) <u>quantum criticality</u>, e.g. density-wave instability, quadrupole nematic order

*) <u>gauge bosons</u> → QED with finite density of fermions
half-filled Landau level
spin liquids

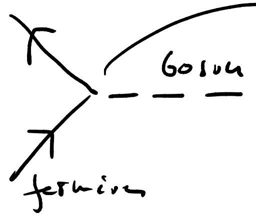
Can we use coupling to Goldstone modes?

<u>No</u>, their interactions ~ q and thus is irrelevant at low energies weak,

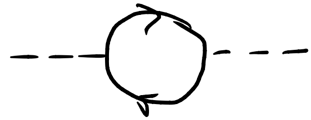
See however Watanabe/Vishwanath 2014 for an exception arxiv:1404.3728

Non-Fermi liquid in 2d (Chubukov, Physics 3, 2010)

Fermions at finite density interact with gapless boson



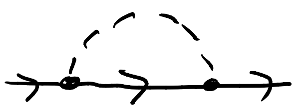
finite coupling constant

Landau damping: 

$$\chi^{-1} = q^2 - i\gamma \frac{\omega}{q}$$

dynamical exponent $z = 3$, $\omega \sim q^z$

Fermionic self-energy in 2d



$$\Sigma(\omega) \sim (1 + i\sqrt{3})\, \omega^{2/3}$$

as $\omega \to 0$ it dominates compared to ω^1 in the inverse bare propagator

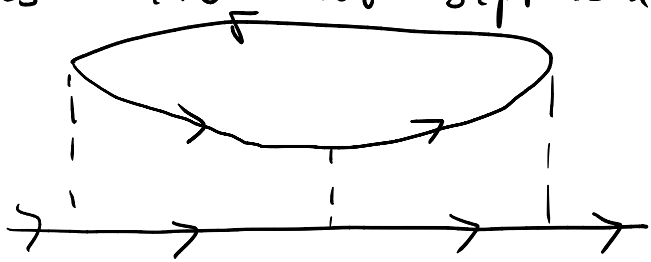
$$G_\psi^{-1} \sim \omega - v_F (k - k_F)$$

and thus both $\epsilon, \Gamma^{-1} \sim (k - k_F)^{3/2}$

quasiparticles do not become sharp at FS, we get NFL behavior

What about higher-loop diagrams?

In attempt to do a well-controlled calculation, people often extend number of fermion species to a large N. For long time people thought that $\omega^{2/3}$ scaling is exact in 2d as $N \to \infty$. But in 2009 Sung-Sik Lee showed that some higher-loop diagrams are not supressed in $1/N$



Metlitski and Sachdev in 2010 showed that $\omega^{2/3}$ is not a correct scaling of the self-energy, they found $\log \omega$ corrections from higher loops. They proposed resummation $\rightsquigarrow$ anomalous dimension

2d NFL problem is still open
see also P. Lee, cond-mat journal club

Another example — QED at finite density of fermions (Holstein et al 1973)

Fermion FS coupled to unscreened transverse el-mag photons

$\Downarrow$

NFL specific heat $\sim T \ln T$

God-given NFL, but is difficult to detect experimentally since α and v_F/c is very small.

Final example — Pomeranchuk instability

$$F_\ell^{a,s} < -1$$

Nematic criticality — transition to the state with spontaneously broken rotation symmetry