Problem set 1: Jordan-Wigner transformation and spin 1/2 XX model

Due date: Oct 30

It was discovered by Jordan and Wigner in 1928 that in one spatial dimension spins 1/2 behave like fermions. In this problem set you are asked to solve the spin 1/2 XX model with the help of the Jordan-Wigner transformation.

1. Spin down and up states can be identified with the empty and singly occupied fermion states

$$|\uparrow\rangle = f^\dagger|0\rangle, \qquad |\downarrow\rangle = |0\rangle. \tag{1}$$

By using the fermion anti-commutation relations show that the operators

$$S^x = \frac{1}{2}(f^\dagger + f), \qquad S^y = \frac{1}{2i}(f^\dagger - f), \qquad S^z = f^\dagger f - \frac{1}{2} \tag{2}$$

satisfy the spin 1/2 commutation relations $[S^a, S^b] = i\epsilon^{abc}S^c$.

2. If one considers a one-dimensional chain of spins, the naive generalization of Eq. (2) does not work because fermions anti-commute on different sites, while spin operators must commute. Jordan and Wigner resolved this issue by expressing the spin at site j as a product of a fermion at that site times the string operator, which is a phase factor that sums over fermion occupancies at all sites to the left of j. If one defines the raising and lowering operators $S_j^\pm = S_j^x \pm iS_j^y$, the Jordan-Wigner transformation is

$$S_j^z = f_j^\dagger f_j - \frac{1}{2}, \qquad S_j^+ = f_j^\dagger e^{i\pi\sum_{l<j} n_l}, \qquad S_j^- = f_j e^{-i\pi\sum_{l<j} n_l}, \tag{3}$$

where $n_l = f_l^\dagger f_l$. Show first that the fermion operator f_j anticommutes with the operator $e^{i\pi n_j}$, i.e., $\{f_j, e^{i\pi n_j}\} = f_j e^{i\pi n_j} + e^{i\pi n_j} f_j = 0$. Using that result, show that spin operators on different sites commute, i.e., demonstrate that $[S_j^{(\pm)}, S_k^{(\pm)}] = 0$ for $j \neq k$.

3. Consider a ferromagnetic spin 1/2 XXZ chain whose Hamiltonian is

$$H = -J\sum_j \left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y\right] - J_z \sum_j S_j^z S_{j+1}^z \tag{4}$$

with $J > 0$. With the help of the Jordan-Wigner transformation (3) show that this Hamiltonian is mapped onto a one-dimensional model of hopping fermions with a nearest neighbor density-density interaction

$$H = -\frac{J}{2}\sum_j \left(f_{j+1}^\dagger f_j + f_j^\dagger f_{j+1}\right) - J_z \sum_j (n_j - \frac{1}{2})(n_{j+1} - \frac{1}{2}). \tag{5}$$

4. Consider a periodic chain with N sites. Using Fourier transformation $f_j = \frac{1}{\sqrt{N}}\sum_k \tilde{f}_k e^{ikj}$, diagonalize the fermionic Hamiltonian (5) at $J_z = 0$. Assuming that all fermionic states with negative energy are occupied in the ground state, what is the ground state magnetization $\langle \sum_j S_j^z \rangle = \langle \sum_j (n_j - 1/2) \rangle$? Determine the dispersion relation ω_k of magnon excitations above the ground state.