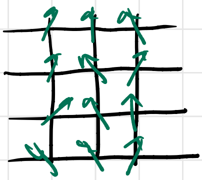
BKT transition

Berezinskii 1972
Kosterlitz, Thouless 1973

Mermin-Wagner, Hohenberg theorem:
No off-diagonal long-range $U(1)$ order (no SSB) is possible in 2d system at finite temperature due to strong low-energy fluctuations due to would-be Goldstone mode

? Does it mean that there is no phase transition between low-temperature and high-temperature regimes?

Consider XY ferromagnet = lattice superfluid



$$H_{XY} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

with $0 < \theta_i < 2\pi$

In 3d system Landau paradigm applies:

$$\lim_{r\to\infty} \langle e^{i[\theta(r)-\theta(0)]} \rangle \sim \begin{cases} \text{const} & T < T_c \\ e^{-r/\xi} & T > T_c \end{cases}$$

Near T_c $\sim r^{\alpha}$ – critical behavior

Consider ~~now~~ the two-dimensional problem and take the continuum limit of H_{XY}:

$$H_{XY} \rightarrow \frac{J}{2} \int d^2r \, (\nabla \theta)^2$$

if we now forget about periodicity of θ and extend $-\infty < \theta < +\infty$, we get the Gaussian model and we find (remember our discussion of Luttinger liquid correlators)

$$\left\langle e^{i[\theta(r) - \theta(0)]} \right\rangle \sim \left(\frac{a}{r}\right)^{\frac{k_B T}{2\pi J}}$$

Paradox: always a power law (CFT like)

no exp.-decaying high-temperature paramagnetic phase like in 3d

BKT realized that there will be a new type of phase transition to the paramagnetic phase driven by unbinding/proliferation of topological defects (quantum vortices)

Main physics: We showed that vortices in 2d superfluids are expensive, effectively cannot be excited.

At low temperature vortex-antivortex pairs of energy $E_{pair} \sim \ln(r_{pair}/\xi)$ are present. As T increases, these bound state pairs unbind for the following reason: since we are at finite temperature, minimize canonical ensemble free energy $F = E - TS$

entropy

For a single vortex we find:

$$E = J\pi \ln(L/\xi) \leftarrow \text{found before}$$

$$S = k_B \ln(L^2/\xi^2)$$

# of configurations where a vortex of size ξ can be placed in a container of size $\sim L$

Energy-entropy balance condition

$$F = J\pi \ln(L/\xi) - T k_B \ln(L^2/\xi^2) = 0$$

$$\Downarrow$$

$$\boxed{T_{KT} = \frac{J\pi}{2k_B}}$$

At $T = T_{KT}$ exciting a free simple vortex costs <u>zero free energy</u> $\Rightarrow$ they will <u>proliferate</u>, i.e., unbind from V-$\bar{V}$ pairs.
There is <u>no spontaneous breaking</u> of $U(1)$ symmetry at $T = T_{KT}$, but a <u>phase transition</u> towards paramagnetic state with exp-decaying correlations.

Systems exhibiting BKT phase transition:

*) 2d superfluid films

*) XY magnet - planar spins

*) 2d superconductors

$$V(r) = \begin{cases} \ln r & r \ll \lambda_p \\ 1/r & r \gg \lambda_p \end{cases} \qquad \text{Pearl length} \quad \lambda_p = \frac{\lambda_L^2}{d}$$

$\lambda_p \gg \lambda_L$ for very thin superconductor films

*) 2d crystals - two separate BKT-like transitions driven by lattice topological defects - dislocations and disclinations.

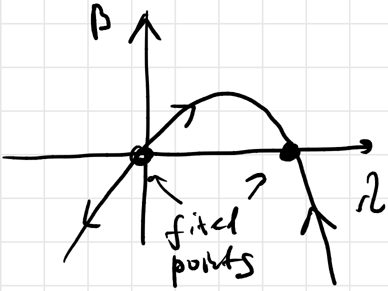
BKT transition can be understood rigorously from <u>renormalization group</u> (RG) arguments, developed by Kosterlitz.

Intro into RG – <u>gradually</u> takes into account (integrate out) fluctuations

running couplings $\quad \partial_t \lambda = \beta(\lambda)$ ← <u>beta function</u>

$$t = \ln(\Lambda/k) \in (0, \infty)$$
UV IR

<u>Usual</u> continuous phase transitions:



close to a fixed point

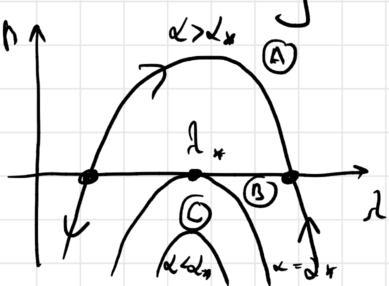
$$\partial_t \delta\lambda = p\, \delta\lambda$$

$$\ln \frac{\delta\lambda}{\delta\lambda_0} = p t$$

$$\delta\lambda(k) = \delta\lambda_0 \left(\frac{\Lambda}{k}\right)^p$$

power-law RG ⇒ critical exponents

BKT scaling – <u>annihilation</u> of two fixed points
Kaplan et al – "Conformality lost"



$$\partial_t \lambda = (\alpha - \alpha_*) - (\lambda - \lambda_*)^2$$

A) $\alpha > \alpha_* \rightarrow$ <u>two fixed points</u>

$$\lambda_\pm = \lambda_* \pm \sqrt{\alpha - \alpha_*}$$

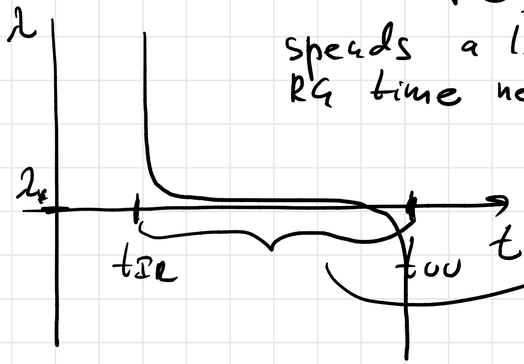
B) $\alpha = \alpha_*$ → two fixed points <u>merge</u>

C) $\alpha < \alpha_*$ → <u>no real fixed points</u>

$$\frac{d\lambda}{dt} = \beta(\lambda) \rightarrow dt = \frac{d\lambda}{\beta(\lambda)} \rightarrow t_{IR} - t_{UV} = \int_{\lambda_{UV}}^{\lambda_{IR}} \frac{d\lambda}{\beta(\lambda)}$$

$$\frac{k_{IR}}{\Lambda} = \exp\left(\int_{\lambda_{UV}}^{\lambda_{IR}} \frac{d\lambda}{\beta(\lambda)}\right) = \exp\left(\left.\frac{\arctan\left(\frac{\lambda - \lambda_*}{\sqrt{\alpha_* - \alpha}}\right)}{\sqrt{\alpha_* - \alpha}}\right|_{\lambda_{UV}}^{\lambda_{IR}}\right)$$

$$\simeq \exp\left(-\frac{\pi}{\sqrt{\alpha_* - \alpha}}\right) \rightsquigarrow \text{insensitive to } \lambda_{IR}, \lambda_{UV}$$



spends a lot of RG time near $\lambda = \lambda_*$

Make now identifications

$k_{IR} \rightsquigarrow \xi^{-1}$ – inverse correlation length

$\alpha \rightarrow T$

$$\boxed{\xi_{BKT}\, T \sim \exp\left(\frac{\#}{\sqrt{T - T_c}}\right)}$$

<u>not</u> a power-law but still <u>non-analytic</u> dependence on $T - T_c$